Option Pricing: A Review

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Outline

Introduction

- These notes review the principles underlying option pricing and some of the key concepts.
- One objective is to highlight the factors that affect option prices, and to see how and why they matter.
- We also discuss important concepts such as the option delta and its properties, implied volatility and the volatility skew.
- For the most part, we focus on the Black-Scholes model, but as motivation and illustration, we also briefly examine the binomial model.

Outline of Presentation

- The material that follows is divided into six (unequal) parts:
  - Options: Definitions, importance of volatility.
  - Pricing of options by replication: Main ideas, a binomial example.
  - The option delta: Definition, importance, behavior.
  - Pricing of options using risk-neutral probabilities.
  - The Black-Scholes model: Assumptions, the formulae, some intuition.
  - Implied Volatility and the volatility skew/smile.

Definitions and Preliminaries

- An *option* is a financial security that gives the holder the right to buy or sell a specified quantity of a specified asset at a specified price on or before a specified date.
  - Buy = *Call* option. Sell = *Put* option
  - On/before: *American*. Only on: *European*
  - Specified price = *Strike* or *exercise* price
  - Specified date = *Maturity* or *expiration* date
  - Buyer = holder = long position
  - Seller = writer = short position

Options as Insurance

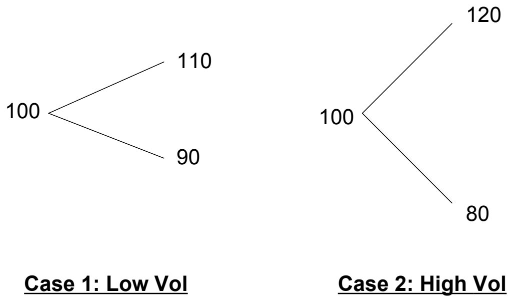
- Options provide *financial insurance*.
  - The option holder has the right, bit not the obligation, to participate in the specified trade.
- Example: Consider holding a put option on Cisco stock with a strike of $25. (Cisco's current price: $26.75.)
  - The put provides a holder of the stock with protection against the price falling below $25.
- What about a call with a strike of (say) $27.50?
  - The call provides a buyer with protection against the price increasing above $25.

The Option Premium

- The writer of the option provides this insurance to the holder.
- In exchange, writer receives an upfront fee called the *option price* or the *option premium*.
- Key question we examine: How is this price determined? What factors matter?

The Importance of Volatility: A Simple Example

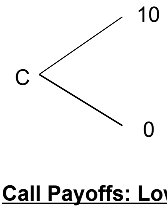
- Suppose current stock price is $S = 100$.
- Consider two possible distributions for S_T. In each case, suppose that the "up" and "down" moves each have probability $1/2$.

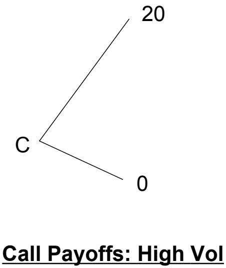


- Same mean but second distribution is more volatile.

Call Payoffs and Volatility

- Consider a call with a strike of $K = 100$.
- Payoffs from the call at maturity:

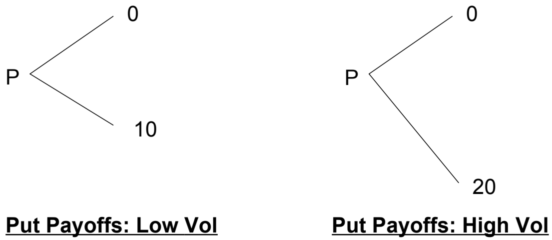




- The second distribution for S_T clearly yields superior payoffs.

Put Payoffs and Volatility

- Puts similarly benefit from volatility. Consider a put with a strike of $K = 100$.
- Payoffs at maturity:



- Once again, the second distribution for S_T clearly yields superior payoffs.

Options and Volatility (Cont'd)

- In both cases, the superior payoffs from high volatility are a consequence of "optionality."
  - A forward with a delivery price of $K = 100$ does not similarly benefit from volatility.
- Thus, all long option positions are also long volatility positions.
  - That is, long option positions increase in value when volatility goes up and decrease in value when volatility goes down.
- Of course, this means that all *written* option positions are short volatility positions.
- Thus, the amount of volatility anticipated over an option's life is a central determinant of option values.

Put–Call Parity

- One of the most important results in all of option pricing theory.
- It relates the prices of otherwise identical European puts and calls:

$$P + S = C + PV(K).$$

- Put-call parity is proved by comparing two portfolios and showing that they have the same payoffs at maturity.
  - **Portfolio A** Long stock, long put with strike K and maturity T.
  - **Portfolio B** Long call with strike K and maturity T, investment of $PV(K)$ for maturity at T.

Options and Replication

- As with all derivatives, the basic idea behind pricing options is *replication*: we look to create identical payoffs to the option's using
  - Long/short positions in the underlying secutiy.
  - Default-risk-free investment/borrowing.
- Once we have a portfolio that replicates the option, the cost of the option must be equal to the cost of replicating (or "synthesizing") it.

Pricing Options by Replication (Cont'd)

- As we have just seen, *volatility* is a primary determinant of option value, so we cannot price options without first modelling volatility.
- More generally, we need to model uncertainty in the evolution of the price of the underlying security.
- It is this dimension that makes option pricing more complex than forward pricing.
- It is also on this dimension that different "option pricing" models make different assumptions:
  - Discrete ("lattice") models: e.g., the binomial.
  - Continuous models: e.g., Black-Scholes.

Pricing Options by Replication (Cont'd)

- Once we have a model of prices evolution, options can be priced by replication:
  - Identify option payoffs at maturity.
  - Set up a portfolio to replicate these payoffs.
  - Value the portfolio and hence price the option.
- The replication process can be technically involved; we illustrate it using a simple example—a one-period binomial model.
- From the example, we draw inferences about the replication process in general, and, in particular, about the behavior of the option *delta*.
- Using the intuition gained here, we examine the Black-Scholes model.

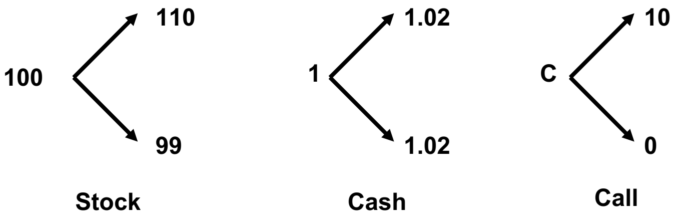
A Binomial Example

- Consider a stock that is currently trading at $S = 100$.
- Suppose that one period from now, it will have one of two possible prices: either $uS = 110$ or $dS = 90$.
- Suppose further that it is possible to borrow or lend over this period at an interest rate of 2%.
- What should be the price of a call option that expires in one period and has a strike price of $K = 100$? What about a similar put option?

Pricing the Call Option

- We price the call in three steps:
  - First, we identify its possible payoffs at maturity.
  - Then, we set up a portfolio to replicate these payoffs.
  - Finally, we compute the cost of this replicating portfolio.
- Step 1 is simple: the call will be exercised if state u occurs, and not otherwise:
  - Call payoff if $uS = 10$.
  - Call payoff if $dS = 0$.

The Call Pricing Problem



The Replicating Portfolio for the Call

- To replicate the call, consider the following portfolio:
  - Δ_c units of stock.
  - B units of lending/borrowing.
- Note that Δ_c and B can be positive or negative.
  - $\Delta_c > 0$: we are *buying* the stock.
  - $\Delta_c < 0$: we are *selling* the stock.
  - $B > 0$: we are *investing*.
  - $B < 0$: we are *borrowing*.

The Replicating Portfolio for the Call

- For the portfolio to replicate the call, we must have:

$$\Delta_c \cdot (110) + B \cdot (1.02) = 10$$

$$\Delta_c \cdot (90) + B \cdot (1.02) = 0$$

- Solving, we obtain:

$$\Delta_C = 0.50 \qquad B = -44.12.$$

- In words, the following portfolio perfectly replicates the call option:
  - A **long** position in 0.50 units of the stock.
  - **Borrowing** of 44.12.

Pricing the Call (Cont'd)

- Cost of this portfolio: $(0.50) \cdot (100) - (44.12)(1) \;=\; 5.88$.
- Since the portfolio perfectly replicates the call, we must have $C = 5.88$.
- Any other price leads to arbitrage:
  - If $C > 5.88$, we can sell the call and buy the replicating portfolio.
  - If $C < 5.88$, we can buy the call and sell the replicating portfolio.

Pricing the Put Option

- To replicate the put, consider the following portfolio:
  - Δ_p units of stock.
  - B units of bond.
- It can be shown that the replicating portfolio now involves a **short** position in the stock and **investment**:
  - $\Delta_p = -0.50$: a short position in 0.50 units of the stock.
  - $B = +53.92$: investment of 53.92.
- As a consequence, the arbitrage-free price of the put is

$$(-0.50)(100) + 53.92 \;=\; 3.92.$$

Summary

- Option prices depend on volatility.
- Thus, option pricing models begin with a description of volatility, or, more generally, of how the prices of the underlying evolves over time.
- Given a model of price evolution, options may be priced by replication.
  - Replicating a call involves a **long** position in the underlying and **borrowing**.
  - Replicating a put involves a **short** position in the underlying and **investment**.
- A key step in the replication process is identification of the option **delta**, i.e., the size of the underlying position in the replicating portfolio. We turn to a more detailed examination of the delta now.

The Option Delta

- The *delta* of an option is the number of units of the underlying security that must be used to replicate the option.
- As such, the delta measures the “riskiness” of the option in terms of the underlying.
- For example: if the delta of an option is (say) $+0.60$, then, roughly speaking, the risk in the option position is the same as the risk in being long 0.60 units of the underlying security.
- Why “roughly speaking?”

The Delta in Hedging Option Risk

- The delta is central to **pricing** options by replication.
- As a consequence, it is also central to **hedging** written option positions.
  - For example, suppose we have written a call whose delta is currently $+0.70$.
  - Then, the risk in the call is the same as the risk in a long position in 0.70 units of the underlying.
  - Since we are **short** the call, we are essentially short 0.70 units of the underlying.
  - Thus, to hedge the position we simply buy 0.70 units of the underlying asset.
  - This is **delta hedging**.

The Delta in Aggregating Option Risk

- The delta enables us to aggregate option risk (on a given underlying) and express it in terms of the underlying.
- For example, suppose we have a portfolio of stocks on IBM stock with possibly different strikes and maturities:
  - Long 2000 calls (strike K_1, maturity T_1), each with a delta of $+0.48$.
  - Long 1000 puts (strike K_2, maturity T_2), each with a delta of -0.55.
  - Short 1700 calls (strike K_3, maturity T_3), each with a delta of $+0.63$.
- What is the aggregate risk in this portfolio?

The Delta in Aggregating Option Risk

- Each of the first group of options (the strike K_1, maturity T_1 calls) is like being **long** 0.48 units of the stock.
- Since the portfolio is long 2,000 of these calls, the aggregate position is akin to being long $2000 \times 0.48 = 960$ units of the stock.
- Similarly, the second group of options (the strike K_2, maturity T_2 puts) is akin to being **short** $1000 \times 0.55 = 550$ units of the stock.
- The third group of options (the strike K_3, maturity T_3 calls) is akin to being **short** $1700 \times 0.63 = 1071$ units of the stock.
- Thus, the aggregate position is: $+960 - 550 - 1071 = -661$ or a **short** position in 661 units of the stock.
- This can be delta hedged by taking an offsetting long position in the stock.

The Delta as a Sensitivity measure

- The delta is also a **sensitivity measure**: it provides a snapshot estimate of the dollar change in the value of a call for a given change in the price of the underlying.
- For example, suppose the delta of a call is $+0.50$.
- Then, holding the call is "like" holding $+0.50$ units of the stock.
- Thus, a change of \$1 in the price of the stock will lead to a change of $+0.50$ in the value of the call.

The Sign of the Delta

- In the binomial examples, the delta of the call was *positive*, while that of the put was *negative*.
- These properties must *always* hold. That is:
  - A long call option position is qualitatively like a long position in the underlying security.
  - A long put option position is qualitatively like a short position in the underlying security.
- Why is this the case?

Maximum Value of the Delta

- Moreover, the delta of a call must always be less than $+1$.
  - The maximum gain in the call's payoff per dollar increase in the price of the underlying is \$1.
  - Thus, we never need more than one unit of the underlying in the replicating portfolio.
- Similarly, the delta of a put must always be greater than -1 since the maximum loss on the put for a \$1 increase in the stock price is \$1.

Depth-in-the-Money ("Moneyness") and the Delta

- The delta of an option depends in a central way on the option's depth-in-the-money.
- Consider a call:
  - If $S \gg K$ (i.e., is very high relative to K), the delta of a call will be close to $+1$.
  - If $S \ll K$ (i.e., is very small relative to K), the delta of the call will be close to zero.
  - In general, as the stock price increases, the delta of the call will increase from 0 to $+1$.

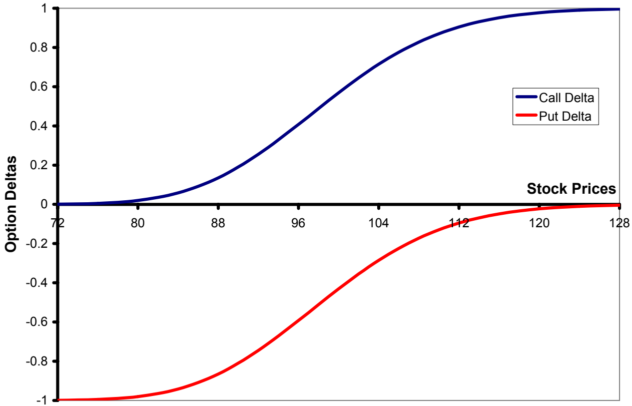
Moneyness and Put Deltas

- Now, consider a put.
  - If $S \gg K$, the put is deep out-of-the-money. Its delta will be close to zero.
  - If $S \ll K$, the put is deep in-the-money. Its delta will be close to -1.
  - In general, as the stock price increases, the delta of the put increases from -1 to 0.

Moneyness and Option Deltas

- To summarize the dependence on moneyness:
  - The delta of a deep out-of-the-money option is close to zero.
  - The absolute value of the delta of a deep in-the-money option is close to 1.
  - As the option moves from out-of-the-money to in-the-money, the absolute value of the delta increases from 0 towards 1.
- The behavior of the call and put deltas are illustrated in the figure on the next page.

The Option Deltas



Implications for Replication/Hedging

- The option delta's behavior has an important implication for option replication.
- In pricing forwards, "buy-and-hold" strategies in the spot asset suffice to replicate the outcome of the forward contract.
- In contrast, since an option's depth-in-the-money changes with time, so will its delta. Thus, a static buy-and-hold strategy will *not* suffice to replicate an option.
- Rather, one must use a *dynamic* replication strategy in which the holding of the underlying security is constantly adjusted to reflect the option's changing delta.

Summary

- The option delta measures the number of units of the underlying that must be held in a replicating portfolio.
- As such, the option delta plays many roles:
  - Replication.
  - (Delta-)Hedging.
  - As a sensitivity measure.
- The option delta depends on depth-in-the-money of the option:
  - It is close to unity for deep in-the-money options.
  - It is close to zero for deep out-of-the-money options.
  - Thus, it offers an intuitive feel for the probability the option will finish in-the-money.

Risk-Neutral Pricing

- An alternative approach to identifying the fair value of an option is to use the model's *risk-neutral* (or *risk-adjusted*) probabilities.
- This approach is guaranteed to give the same answer as replication, but is computationally a lot simpler.
- Pricing follows a three-step procedure.
    - Identify the model's risk-neutral probability.
    - Take expectations of the option's payoffs under the risk-neutral probability.
    - Discount these expectations back to the present at the risk-free rate.
- The number obtained in Step 3 is the fair value of the option.

The Risk-Neutral Probability

- The risk-neutral probability is that probability under which expected returns on all the model's assets are the same.
- For example, the binomial model has two assets: the stock which returns u or d, and the risk-free asset which returns R. ($R = 1+$ the risk-free interest rate.)
- If q and $1-q$ denote, respectively, the risk-neutral probabilities of state u and state d, then q must satisfy

$$q \cdot u + (1-q) \cdot d \ = \ R.$$

- This means the risk-neutral probability in the binomial model is

$$q \ = \ \frac{R-d}{u-d}.$$

Risk-Neutral Pricing: An Example

- Consider the one-period binomial example in which $u = 1.10$, $d = 0.90$, and $R = 1.02$.
- In this case,

$$q = \frac{1.02 - 0.90}{1.10 - 0.90} = 0.60.$$

- Therefore, the expected payoffs of the call under the risk-neutral probability is

$$(0.60)(10) + (0.40)(0) = 6.$$

- Discounting these payoffs back to the present at the risk-free rate results in

$$\frac{6}{1.02} = 5.88,$$

which is the same price for the option obtained by replication.

Risk-Neutral Pricing: A Second Example

- Now consider pricing the put with strike $K = 100$.
- The expected payoff from the put at maturity is

$$(0.60)(0) + (0.40)(10) \;=\; 4.$$

- Discounting this back to the present at the risk-free rate, we obtain

$$\frac{4}{1.02} \;=\; 3.92,$$

which is the same price obtained via replication.

- As these examples show, risk-neutral pricing is computationally much simpler than replication.

The Balck-Scholes Model

- The Black–Scholes model is unambiguously the best known model of option pricing.
- Also one of the most widely used: it is the benchmark model for
  - Equities.
  - Stock indices.
  - Currencies.
  - Futures.
- Moreover, it forms the basis of the Black model that is commonly used to price some interest-rate derivatives such as caps and floors.

The Black-Scholes Model (Cont'd)

- Technically, the Black-Scholes model is much more complex than discrete models like the binomial.
  - In particular, it assumes *continuous* evolution of uncertainty.
  - Pricing options in this framework requires the use of very sophisticated mathematics.
- What is gained by all this sophistication?
  - Option prices in the Black-Scholes model can be expressed in *closed-form*, i.e., as particular explicit functions of the parameters.
  - This makes computing option prices and option sensitivities very easy.

Assumptions of the Model

- The main assumption of the Black-Scholes model pertains to the evolution of the stock price.
- This price is taken to evolve according to a **geometric Brownian motion**.
- Shorn of technical details, this says essentially that two conditions must be met:
  - Returns on the stock have a **lognormal** distribution with constant volatility.
  - Stock prices cannot jump (the market cannot "gap").

The Log-Normal Assumption

- The log-normal assumption says that the (natural) **log** of returns is **normally** distributed: if S denotes the current price, and S_t the price t years from the present, then

$$\ln\left(\frac{S_t}{S}\right) \sim N(\mu t, \sigma^2 t).$$

- Mathematically, **log-returns** and **continuously-compounded returns** represent the same thing:

$$\ln\left(\frac{S_t}{S}\right) = x \quad \Leftrightarrow \quad \frac{S_t}{S} = e^x \quad \Leftrightarrow \quad S_t = Se^x.$$

- Thus, log-normality says that returns on the stock, expressed in continuously-compounded terms, are normally distributed.

Volatility

- The number σ is called the *volatility* of the stock. Thus, volatility in the Black-Scholes model refers to the standard deviation of annual log-returns.
- The Black-Scholes model takes this volatility to be a constant. In principle, this volatility can be estimated in two ways:
    - From historical data. (This is called *historical volatility*.)
    - From options prices. (This is called *implied volatility*.)
- We discuss the issue of volatility estimation in the last part of this segment. For now, assume the level of volatility is known.

Is GBM a Reasonable Assumption?

- The log-normality and no-jumps conditions appear unreasonably restrictive:
  - Volatility of markets is typically not constant over time.
  - Market prices do sometimes "jump."
  - In particular, the no-jumps assumption appears to rule out dividends.
- Dividends (and similar predictable jumps) are actually easily handled by the model.
- The other issues are more problematic, and not easily resolved. We discuss them in the last part of this presentation.

Option Prices in the Black-Scholes Model

- Option prices in the Black–Scholes model may be recovered using a replicating portfolio argument.
- Of course, the construction—and maintenence—of a replicating portfolio is significantly more technically complex here than in the binomial model.
- We focus here instead on the final option prices that result and the intuitive content of these prices.

Notation

- t: current time.
- T: Horizon of the model. (So time-left-to-maturity: $T - t$.)
- K: strike price of option.
- S_t: current price of stock.
- S_T: stock price at T.
- μ, σ: Expected return and volatility of stock (annualized).
- r: risk-free rate of interest.
- C, P: Prices of call and put (European only).

Call Option Prices in the Black-Scholes Model

- The call-pricing formula in the Black-Scholes model is

$$C = S_t \cdot N(d_1) - e^{-r(T-t)} K \cdot N(d_2)$$

where $N(\cdot)$ is the *cumulative* standard normal distribution [$N(x)$ is the probability under a standard normal distribution of an observation less than or equal to x], and

$$d_1 = \frac{1}{\sigma\sqrt{T-t}} \left[\ln\left(\frac{S_t}{K}\right) + (r + \frac{1}{2}\sigma^2)(T-t) \right]$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

- This formula has a surprisingly simple interpretation.

Call Prices and Replication

- To replicate a call in general, we must
  - Take a long position in Δ_c units of the underlying, and
  - Borrow B_c at the risk-free rate.
- The cost of this replicating portfolio—which is the call price—is

$$C = S_t \cdot \Delta_c - B_c. \qquad (1)$$

- The Black-Scholes formula has an identical structure: it too is of the form

$$C = S_t \times [\text{Term 1}] - [\text{Term 2}]. \qquad (2)$$

Call Prices and Replication (Cont'd)

- Comparing these structures suggests that

$$\Delta_c = N(d_1). \tag{3}$$

$$B_c = e^{-r(T-t)} K \cdot N(d_2). \tag{4}$$

- This is exactly correct! The Black-Scholes formula is obtained precisely by showing that the composition of the replicating portfolio is (3)–(4) and substituting this into (1).
- In particular, $N(d_1)$ is just the delta of the call option.

Put Option Prices in the Black–Scholes Model

- Recall that to replicate a put in general, we must
  - Take a *short* position in $|\Delta_p|$ units of the underlying, and
  - Lend B_p at the risk-free rate.
- Thus, in general, we can write the price of the put as

$$P = B_p + S_t, \Delta_p \tag{5}$$

- The Black-Scholes formula identifies the exact composition of Δ_p and B_p:

$$\Delta_p = -N(-d_1) \qquad B_p = PV(K)\, N(-d_2)$$

where $N(\cdot)$, d_1, and d_2 are all as defined above.

Put Prices in the Black–Scholes Model (Cont'd)

- Therefore, the price of the put is given by

$$P = PV(K) \cdot N(-d_2) - S_t \cdot N(-d_1). \qquad (6)$$

- Expression (6) is the Black-Scholes formula for a European put option.
- Equivalently, since $N(x) + N(-x) = 1$ for any x, we can also write

$$P = S_t \cdot [N(d_1) - 1] + PV(K) \cdot [1 - N(d_2)] \qquad (7)$$

Black-Scholes via Risk-Neutral Probabilities

- Alternative way to derive Black–Scholes formulae: use risk-neutral (or risk-adjusted) probabilities.
- To identify option prices in this approach: take expectation of terminal payoffs under the risk-neutral probability measure and discount at the risk-free rate.
- The terminal payofffs of a call option with strike K are given by

$$\max\{S_T - K, 0\}.$$

- Therefore, the arbitrage-free price of the call option is given by

$$C = e^{-rT} E_t^* [\max\{S_T - K, 0\}]$$

where $E_t^*[\cdot]$ denotes time–t expectations under the risk-neutral measure.

Black-Scholes via Risk-Neutral Probs (Cont'd)

- Equivalently, this expression may be written as

$$C = e^{-rT} E_t^* [S_T - K \,||\, S_T \geq K]$$

- Splitting up the expectation, we have

$$C = e^{-rT} E_t^* [S_T \,||\, S_T \geq K] - e^{-rT} E_t^* [K \,||\, S_T \geq K].$$

- From this to the Black–Scholes formula is simply a matter of grinding through the expectations, which are tedious, but not otherwise difficult.

Black-Scholes via Risk-Neutral Probs (Cont'd)

- Specifically, it can be shown that

$$e^{-rT}\, E_t^*\,[S_T \,||\, S_T \geq K] \;=\; S_t\, N(d_1)$$

$$e^{-rT}\, E_t^*\,[K \,||\, S_T \geq K] \;=\; e^{-rT} K\, N(d_2)$$

- In particular, $N(d_2)$ is the risk-neutral probability that the option finishes in-the-money (i.e., that $S_T \geq K$).

Remarks on the Black-Scholes Formulae

- Two remarkable features of the Black–Scholes formulae:
  - Option prices only depend on *five* variables: S, K, r, T, and σ.
  - Of these five variables, only *one*—the volatility σ—is not directly observable.
- This makes the model easy to implement in practice.

Remarks (Cont'd)

- It must also be stressed that these are *arbitrage-free* prices.
  - That is, they are based on construction of replicating portfolios that perfectly mimic option payoffs at maturity.
  - Thus, if prices differ from these predicted levels, the replicating portfolios can be used to create riskless profits.
- The formulae can also be used to delta-hedge option positions.
  - For example, suppose we have written a call option whose current delta, using the Black-Scholes formula, is $N(d_1)$.
  - To hedge this position, we take a long position in $N(d_1)$ units of the underlying.
  - Of course, dynamic hedging is required.

Remarks (Cont'd)

- Finally, it must be stressed again that closed-form expressions of this sort for option prices is a rare occurence.
  - The particular advantage of having closed forms is that they make computation of option sensitivities (or option "greeks") a simple task.
- Nonetheless, such closed-form expressions exist in the Black-Scholes framework only for *European*-style options.
  - For example, closed-forms do not exist for American put options.
  - However, it *is* possible to obtain closed-form solutions for certain classes of exotic options (such as compound options or barrier options).

Plotting Black-Scholes Option Prices

- Closed-forms make it easy to compute option prices in the Black-Scholes model:

  1. Input values for S_t, K, r, $T-t$, and σ.
  2. Compute $d_1 = [\ln(S_t/K) + (r+\sigma^2/2)(T-t)]/[\sigma\sqrt{T-t}]$.
  3. Compute $d_2 = d_1 - \sigma\sqrt{T-t}$.
  4. Compute $N(d_1)$.
  5. Compute $N(d_2)$.
  6. Compute option prices.

$$C = S_t N(d_1) - e^{-r(T-t)} K\, N(d_2)$$

$$P = e^{-r(T-t)} K\, [1 - N(d_2)] - S_t\, [1 - N(d_1)]$$

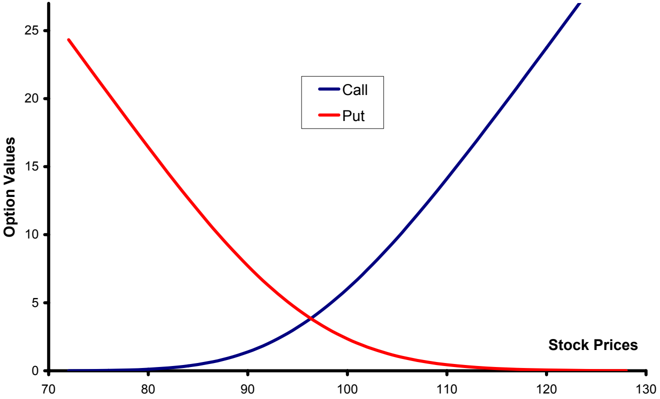
Plotting Option Prices (Cont'd)

- The following figure illustrates this procedure.
- Four parameters are held fixed in the figure:

$$K = 100, \; T - t = 0.50, \sigma = 0.20, r = 0.05$$

- The figures plot call and put prices as S varies from 72 to 128.
- Observe non-linear reaction of option prices to changes in stock price.
  - For deep OTM options, slope ≈ 0.
  - For deep ITM options, slope $\approx$1.
- Of course, this slope is precisely the option delta!

Plotting Option Prices



Implied Voaltility

- Given an option price, one can ask the question: what level of volatility is implied by the observed price?
- This level is the *implied volatility*.
- Formally, implied volatility is the volatility level that would make observed option prices consistent with the Black-Scholes formula, given values for the other parameters.

Implied Volatility (Cont'd)

- For example, suppose we are looking at a call on a non-dividend-paying stock.
- Let K and $T-t$ denote the call's strike and time-to-maturity, and let $\widehat{C}$ be the call's price.
- Let S_t be the stock price and r the interest rate.
- Then, the implied volatility is the unique level σ for which

$$C^{bs}(S, K, T-t, r, \sigma) \;=\; \widehat{C},$$

  where C^{bs} is the Black-Scholes call option pricing formula.
- Note that implied volatility is uniquely defined since C^{bs} is strictly increasing in σ.

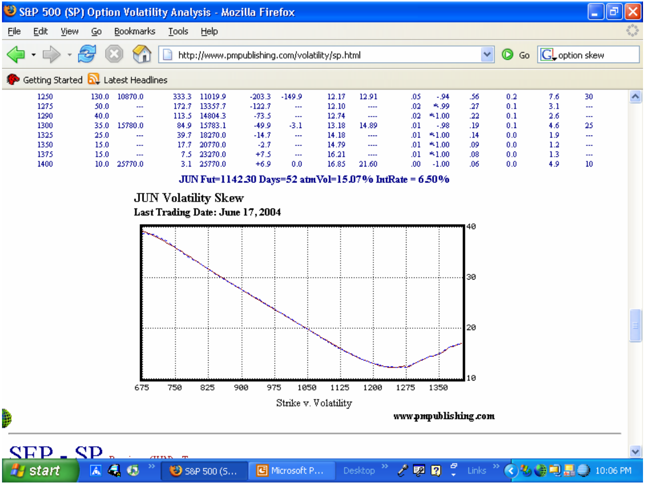
Implied Volatility (Cont'd)

- Implied volatility represents the "market's" perception of volatility anticipated over the option's lifetime.
- Implied volatility is thus *forward looking*.
- In contrast, historical volatility is *backward looking*.

The Volatility Smile/Skew

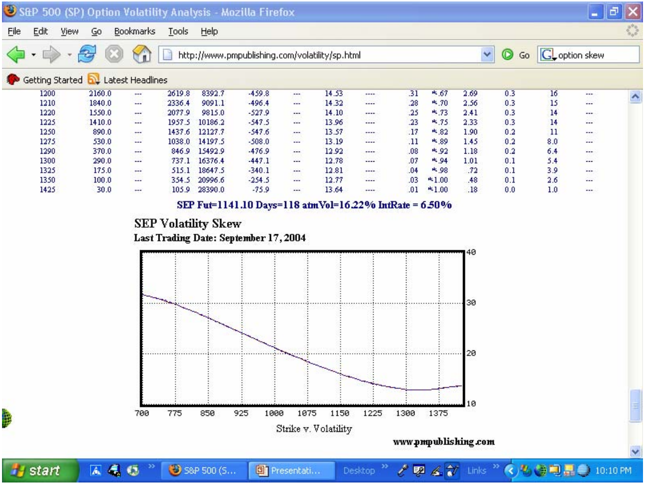
- In theory, any option (any K or T) may be used for measuring implied volatility.
- Thus, if we fix maturity and plot implied volatilities against strike prices, the plot should be a flat line.
- In practice, in equity markets, implied volatilities for "low" strikes (corresponding to out-of-the-money puts) are typically much higher than implied volatilties for ATM options. This is the *volatility skew*.
- In currency markets (and for many individual equities), the picture is more symmetric with way-from-the-money options having higher implied volatilities than at-the-money options. This is the *volatility smile*.
- See the screenshots on the next 4 slides.

S&P 500 Implied Voaltility Plot

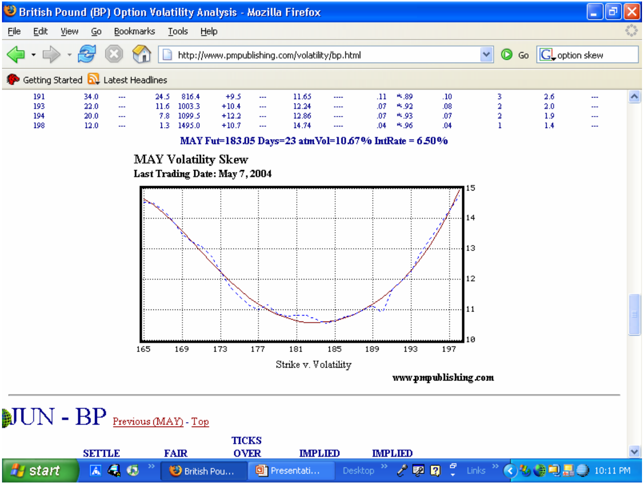


1250	130.0	10870.0	333.3	11019.9	-203.3	-149.9	12.17	12.91	.05	-.94	.56	0.2	7.6	30
1275	50.0	---	172.7	13357.7	-122.7	---	12.10	----	.02	*.99	.27	0.1	3.1	---
1290	40.0	---	113.5	14804.3	-73.5	---	12.74	----	.02	*1.00	.22	0.1	2.6	---
1300	35.0	15780.0	84.9	15783.1	-49.9	-3.1	13.18	14.89	.01	-.98	.19	0.1	4.6	25
1325	25.0	---	39.7	18270.0	-14.7	---	14.18	----	.01	*1.00	.14	0.0	1.9	---
1350	15.0	---	17.7	20770.0	-2.7	---	14.79	----	.01	*1.00	.09	0.0	1.2	---
1375	15.0	---	7.5	23270.0	+7.5	---	16.21	----	.01	*1.00	.08	0.0	1.3	---
1400	10.0	25770.0	3.1	25770.0	+6.9	0.0	16.85	21.60	.00	-1.00	.06	0.0	4.9	10

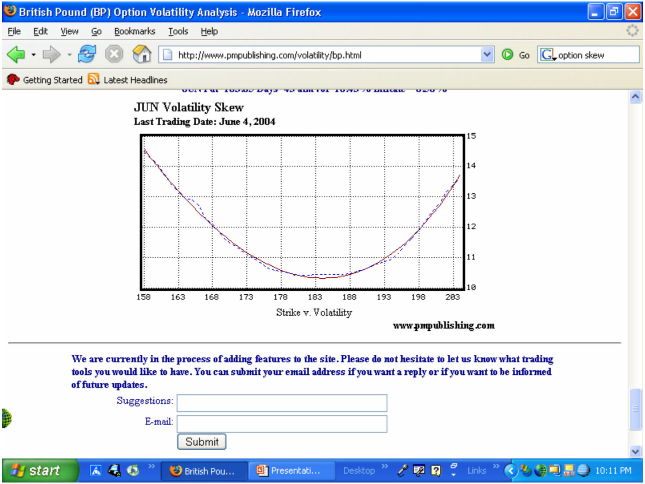
S&P 500 Implied Voaltility Plot



USD-GBP Implied Volatility Plot



USD-GBP Implied Volatility Plot



The Source of the Volatility Skew

- Two sources are normally ascribed for the volatility skew.
- One is the returns distribution. The Black-Scholes model assumes log-returns are normally distributed.
- However, in *every* financial market, extreme observations are far more likely than predicted by the log-normal distribution.
    - Extreme observations = observations in the *tail* of the distribution.
    - Empirical distributions exhibit "fat tails" or *leptokurtosis*.
- Empirical log-returns distributions are often also *skewed*.

Source of the Volatility Skew (Cont'd)

- Fat tails $\Longrightarrow$ Black-Scholes model with a constant volatility will *underprice* out-of-the-money puts relative to those at-the-money.
- Put differently, "correctly" priced out-of-the-money puts will reflect a higher implied volatility than at-themoney options.
- This is exactly the volatility skew! That is, the volatility skew is evidence not only that the lognormal model is not a fully accurate description of reality but also that the market recognizes this shortcoming.
- As such, there is valuable information in the smile/skew concerning the "actual" (more accurately, the market's expectation) of the return distribution. For instance:
  - More symmetric smile $\Longrightarrow$ Less skewed distribution.
  - Flatter smile/skew $\Longrightarrow$ Smaller kurtosis.

Source of the Volatility Skew (Cont'd)

- The other reason commonly given for the volatility skew is that the world as a whole is net long equities, and so there is a positive net demand for protection in the form of puts.
- This demand for protection, coupled with market frictions, raises the price of out-of-the-money puts relative to those at-the-money, and results in the volatility skew.
- With currencies, on the other hand, there is greater symmetry since the world is net long both currencies, so there is two-sided demand for protection.
- Implicit in this argument is Rubinstein's notion of "crash-o-phobia," fear of a sudden large downward jump in prices.

Generalizing Black-Scholes

- Obvious question: why not generalize the log-normal distribution?
- Indeed, there may even be a "natural" generalization.
- The Black-Scholes model makes two uncomfortable assumptions:
  - No jumps.
  - Constant volatility.
- If jumps are added to the log-normal model or if volatility is allowed to be stochastic, the model will exhibit fat tails and even skewness.

Generalizing Black-Scholes (Cont'd)

- So why don't we just do this?
  - We can, but complexity increases (how many new parameters do we need to estimate?).
  - Jumps & SV have very different dynamic implications.
  - Reality is more complex than either model (indeed, substantially so).
- Ultimately: better to use a simple model with known shortcomings?
- On jumps vs stochastic volatility models, see

  Das, S.R. and R.K. Sundaram (1999) "Of Smiles and Smirks: A Term-Structure Perspective," Journal of Financial and Quantitative Analysis *34(2), pp.211-239.*