Option Pricing: A Review

Rangarajan K. Sundaram

Stern School of Business
New York University

Invesco Great Wall Fund Management Co.
Shenzhen: June 14, 2008
Outline

1 Introduction

2 Pricing Options by Replication

3 The Option Delta

4 Option Pricing using Risk-Neutral Probabilities

5 The Black-Scholes Model

6 Implied Volatility
These notes review the principles underlying option pricing and some of the key concepts.

One objective is to highlight the factors that affect option prices, and to see how and why they matter.

We also discuss important concepts such as the option delta and its properties, implied volatility and the volatility skew.

For the most part, we focus on the Black-Scholes model, but as motivation and illustration, we also briefly examine the binomial model.
The material that follows is divided into six (unequal) parts:

- Options: Definitions, importance of volatility.
- Pricing of options by replication: Main ideas, a binomial example.
- The option delta: Definition, importance, behavior.
- Pricing of options using risk-neutral probabilities.
- The Black-Scholes model: Assumptions, the formulae, some intuition.
- Implied Volatility and the volatility skew/smile.
Definitions and Preliminaries

An option is a financial security that gives the holder the right to buy or sell a specified quantity of a specified asset at a specified price on or before a specified date.

- Buy = Call option. Sell = Put option
- On/before: American. Only on: European
- Specified price = Strike or exercise price
- Specified date = Maturity or expiration date
- Buyer = holder = long position
- Seller = writer = short position
Options as Insurance

- Options provide *financial insurance*.
  - The option holder has the right, but not the obligation, to participate in the specified trade.

- Example: Consider holding a put option on Cisco stock with a strike of $25. (Cisco's current price: $26.75.)
  - The put provides a holder of the stock with protection against the price falling below $25.

- What about a call with a strike of (say) $27.50?
  - The call provides a buyer with protection against the price increasing above $25.
The Option Premium

- The writer of the option provides this insurance to the holder.
- In exchange, writer receives an upfront fee called the *option price* or the *option premium*.
- Key question we examine: How is this price determined? What factors matter?
The Importance of Volatility: A Simple Example

- Suppose current stock price is \( S = 100 \).
- Consider two possible distributions for \( S_T \). In each case, suppose that the “up” and “down” moves each have probability \( 1/2 \).

![Diagram]

**Case 1: Low Vol**
- \( S_T \) can be 110 or 90.

**Case 2: High Vol**
- \( S_T \) can be 120 or 80.

- Same mean but second distribution is more volatile.
Consider a call with a strike of $K = 100$.

Payoffs from the call at maturity:

- **Call Payoffs: Low Vol**
  - Payoffs: $10 - 0$

- **Call Payoffs: High Vol**
  - Payoffs: $20 - 0$

The second distribution for $S_T$ clearly yields superior payoffs.
Puts similarly benefit from volatility. Consider a put with a strike of $K = 100$.

Payoffs at maturity:

- **Put Payoffs: Low Vol**
  - Payoff: $0$
  - Payoff: $10$

- **Put Payoffs: High Vol**
  - Payoff: $0$
  - Payoff: $20$

Once again, the second distribution for $S_T$ clearly yields superior payoffs.
In both cases, the superior payoffs from high volatility are a consequence of “optionality.”

- A forward with a delivery price of $K = 100$ does not similarly benefit from volatility.

- Thus, all **long** option positions are also **long volatility** positions.

- That is, long option positions increase in value when volatility goes up and decrease in value when volatility goes down.

- Of course, this means that all **written** option positions are **short volatility** positions.

- Thus, the amount of volatility anticipated over an option’s life is a central determinant of option values.
Put–Call Parity

- One of the most important results in all of option pricing theory.
- It relates the prices of otherwise identical European puts and calls:

\[ P + S = C + PV(K). \]

- Put-call parity is proved by comparing two portfolios and showing that they have the same payoffs at maturity.
  - **Portfolio A** Long stock, long put with strike \( K \) and maturity \( T \).
  - **Portfolio B** Long call with strike \( K \) and maturity \( T \), investment of \( PV(K) \) for maturity at \( T \).
As with all derivatives, the basic idea behind pricing options is replication: we look to create identical payoffs to the option’s using

- Long/short positions in the underlying security.
- Default-risk-free investment/borrowing.

Once we have a portfolio that replicates the option, the cost of the option must be equal to the cost of replicating (or “synthesizing”) it.
As we have just seen, *volatility* is a primary determinant of option value, so we cannot price options without first modelling volatility.

More generally, we need to model uncertainty in the evolution of the price of the underlying security.

It is this dimension that makes option pricing more complex than forward pricing.

It is also on this dimension that different “option pricing” models make different assumptions:

- Discrete ("lattice") models: e.g., the binomial.
- Continuous models: e.g., Black-Scholes.
Once we have a model of prices evolution, options can be priced by replication:

- Identify option payoffs at maturity.
- Set up a portfolio to replicate these payoffs.
- Value the portfolio and hence price the option.

The replication process can be technically involved; we illustrate it using a simple example—a one-period binomial model.

From the example, we draw inferences about the replication process in general, and, in particular, about the behavior of the option $delta$.

Using the intuition gained here, we examine the Black-Scholes model.
Consider a stock that is currently trading at $S = 100$.

Suppose that one period from now, it will have one of two possible prices: either $uS = 110$ or $dS = 90$.

Suppose further that it is possible to borrow or lend over this period at an interest rate of 2%.

What should be the price of a call option that expires in one period and has a strike price of $K = 100$? What about a similar put option?
Pricing the Call Option

- We price the call in three steps:
  - First, we identify its possible payoffs at maturity.
  - Then, we set up a portfolio to replicate these payoffs.
  - Finally, we compute the cost of this replicating portfolio.

- Step 1 is simple: the call will be exercised if state $u$ occurs, and not otherwise:
  - Call payoff if $uS = 10$.
  - Call payoff if $dS = 0$. 
The Call Pricing Problem

<table>
<thead>
<tr>
<th>Stock</th>
<th>Cash</th>
<th>Call</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>1</td>
<td>C</td>
</tr>
<tr>
<td>99</td>
<td>1.02</td>
<td>10</td>
</tr>
<tr>
<td>110</td>
<td>1.02</td>
<td>0</td>
</tr>
</tbody>
</table>

Diagram:
- Stock: 100 → 99, 110
- Cash: 1 → 1.02, 1.02
- Call: C → 10, 0
To replicate the call, consider the following portfolio:

- $\Delta_c$ units of stock.
- $B$ units of lending/borrowing.

Note that $\Delta_c$ and $B$ can be positive or negative.

- $\Delta_c > 0$: we are buying the stock.
- $\Delta_c < 0$: we are selling the stock.
- $B > 0$: we are investing.
- $B < 0$: we are borrowing.
The Replicating Portfolio for the Call

For the portfolio to replicate the call, we must have:

\[ \Delta_c \cdot (110) + B \cdot (1.02) = 10 \]

\[ \Delta_c \cdot (90) + B \cdot (1.02) = 0 \]

Solving, we obtain:

\[ \Delta_c = 0.50 \quad B = -44.12. \]

In words, the following portfolio perfectly replicates the call option:

- A **long** position in 0.50 units of the stock.
- **Borrowing** of 44.12.
Pricing the Call (Cont’d)

- Cost of this portfolio: \((0.50) \cdot (100) - (44.12)(1) = 5.88\).
- Since the portfolio perfectly replicates the call, we must have \(C = 5.88\).
- Any other price leads to arbitrage:
  - If \(C > 5.88\), we can sell the call and buy the replicating portfolio.
  - If \(C < 5.88\), we can buy the call and sell the replicating portfolio.
Pricing the Put Option

To replicate the put, consider the following portfolio:

- $\Delta_p$ units of stock.
- $B$ units of bond.

It can be shown that the replicating portfolio now involves a short position in the stock and investment:

- $\Delta_p = -0.50$: a short position in 0.50 units of the stock.
- $B = +53.92$: investment of 53.92.

As a consequence, the arbitrage-free price of the put is

$$(-0.50)(100) + 53.92 = 3.92.$$
Summary

- Option prices depend on volatility.
- Thus, option pricing models begin with a description of volatility, or, more generally, of how the prices of the underlying evolves over time.
- Given a model of price evolution, options may be priced by replication.
  - Replicating a call involves a long position in the underlying and borrowing.
  - Replicating a put involves a short position in the underlying and investment.
- A key step in the replication process is identification of the option delta, i.e., the size of the underlying position in the replicating portfolio. We turn to a more detailed examination of the delta now.
The Option Delta

- The *delta* of an option is the number of units of the underlying security that must be used to replicate the option.

- As such, the delta measures the “riskiness” of the option in terms of the underlying.

- For example: if the delta of an option is (say) +0.60, then, roughly speaking, the risk in the option position is the same as the risk in being long 0.60 units of the underlying security.

- Why “roughly speaking?”
The Delta in Hedging Option Risk

- The delta is central to **pricing** options by replication.
- As a consequence, it is also central to **hedging** written option positions.

  - For example, suppose we have written a call whose delta is currently +0.70.
  - Then, the risk in the call is the same as the risk in a long position in 0.70 units of the underlying.
  - Since we are short the call, we are essentially short 0.70 units of the underlying.
  - Thus, to hedge the position we simply buy 0.70 units of the underlying asset.
  - This is **delta hedging**.
The Delta in Aggregating Option Risk

- The delta enables us to aggregate option risk (on a given underlying) and express it in terms of the underlying.

- For example, suppose we have a portfolio of stocks on IBM stock with possibly different strikes and maturities:
  - Long 2000 calls (strike $K_1$, maturity $T_1$), each with a delta of $+0.48$.
  - Long 1000 puts (strike $K_2$, maturity $T_2$), each with a delta of $-0.55$.
  - Short 1700 calls (strike $K_3$, maturity $T_3$), each with a delta of $+0.63$.

- What is the aggregate risk in this portfolio?
The Delta in Aggregating Option Risk

- Each of the first group of options (the strike $K_1$, maturity $T_1$ calls) is like being long 0.48 units of the stock.

- Since the portfolio is long 2,000 of these calls, the aggregate position is akin to being long $2000 \times 0.48 = 960$ units of the stock.

- Similarly, the second group of options (the strike $K_2$, maturity $T_2$ puts) is akin to being short $1000 \times 0.55 = 550$ units of the stock.

- The third group of options (the strike $K_3$, maturity $T_3$ calls) is akin to being short $1700 \times 0.63 = 1071$ units of the stock.

- Thus, the aggregate position is: $+960 - 550 - 1071 = -661$ or a short position in 661 units of the stock.

- This can be delta hedged by taking an offsetting long position in the stock.
The Delta as a Sensitivity measure

- The delta is also a **sensitivity measure**: it provides a snapshot estimate of the dollar change in the value of a call for a given change in the price of the underlying.

- For example, suppose the delta of a call is $+0.50$.

- Then, holding the call is “like” holding $+0.50$ units of the stock.

- Thus, a change of $1$ in the price of the stock will lead to a change of $+0.50$ in the value of the call.
The Sign of the Delta

- In the binomial examples, the delta of the call was positive, while that of the put was negative.

- These properties must always hold. That is:
  - A long call option position is qualitatively like a long position in the underlying security.
  - A long put option position is qualitatively like a short position in the underlying security.

- Why is this the case?
Maximum Value of the Delta

- Moreover, the delta of a call must always be less than +1.
  - The maximum gain in the call’s payoff per dollar increase in the price of the underlying is $1.
  - Thus, we never need more than one unit of the underlying in the replicating portfolio.

- Similarly, the delta of a put must always be greater than −1 since the maximum loss on the put for a $1 increase in the stock price is $1.
The delta of an option depends in a central way on the option’s depth-in-the-money.

Consider a call:

- If $S \gg K$ (i.e., is very high relative to $K$), the delta of a call will be close to $+1$.
- If $S \ll K$ (i.e., is very small relative to $K$), the delta of the call will be close to zero.
- In general, as the stock price increases, the delta of the call will increase from $0$ to $+1$. 
Moneyness and Put Deltas

Now, consider a put.

- If $S \gg K$, the put is deep out-of-the-money. Its delta will be close to zero.
- If $S \ll K$, the put is deep in-the-money. Its delta will be close to $-1$.
- In general, as the stock price increases, the delta of the put increases from $-1$ to $0$. 
Moneyness and Option Deltas

- To summarize the dependence on moneyness:
  - The delta of a deep out-of-the-money option is close to zero.
  - The absolute value of the delta of a deep in-the-money option is close to 1.
  - As the option moves from out-of-the-money to in-the-money, the absolute value of the delta increases from 0 towards 1.
- The behavior of the call and put deltas are illustrated in the figure on the next page.
The Option Deltas

- Call Delta
- Put Delta

Stock Prices

The Option Delta

Option Pricing using Risk-Neutral Probabilities

The Black-Scholes Model

Implied Volatility
The option delta’s behavior has an important implication for option replication.

In pricing forwards, “buy-and-hold” strategies in the spot asset suffice to replicate the outcome of the forward contract.

In contrast, since an option's depth-in-the-money changes with time, so will its delta. Thus, a static buy-and-hold strategy will not suffice to replicate an option.

Rather, one must use a dynamic replication strategy in which the holding of the underlying security is constantly adjusted to reflect the option’s changing delta.
Summary

- The option delta measures the number of units of the underlying that must be held in a replicating portfolio.

- As such, the option delta plays many roles:
  - Replication.
  - (Delta-)Hedging.
  - As a sensitivity measure.

- The option delta depends on depth-in-the-money of the option:
  - It is close to unity for deep in-the-money options.
  - It is close to zero for deep out-of-the-money options.
  - Thus, it offers an intuitive feel for the probability the option will finish in-the-money.
Risk-Neutral Pricing

- An alternative approach to identifying the fair value of an option is to use the model’s risk-neutral (or risk-adjusted) probabilities.
- This approach is guaranteed to give the same answer as replication, but is computationally a lot simpler.
- Pricing follows a three-step procedure.
  - Identify the model’s risk-neutral probability.
  - Take expectations of the option’s payoffs under the risk-neutral probability.
  - Discount these expectations back to the present at the risk-free rate.
- The number obtained in Step 3 is the fair value of the option.
The risk-neutral probability is that probability under which expected returns on all the model’s assets are the same.

For example, the binomial model has two assets: the stock which returns $u$ or $d$, and the risk-free asset which returns $R$. ($R = 1 + \text{the risk-free interest rate}$.)

If $q$ and $1 - q$ denote, respectively, the risk-neutral probabilities of state $u$ and state $d$, then $q$ must satisfy

$$q \cdot u + (1 - q) \cdot d = R.$$ 

This means the risk-neutral probability in the binomial model is

$$q = \frac{R - d}{u - d}.$$
Risk-Neutral Pricing: An Example

- Consider the one-period binomial example in which \( u = 1.10 \), \( d = 0.90 \), and \( R = 1.02 \).
- In this case,
  \[
  q = \frac{1.02 - 0.90}{1.10 - 0.90} = 0.60.
  \]
- Therefore, the expected payoffs of the call under the risk-neutral probability is
  \[
  (0.60)(10) + (0.40)(0) = 6.
  \]
- Discounting these payoffs back to the present at the risk-free rate results in
  \[
  \frac{6}{1.02} = 5.88,
  \]
  which is the same price for the option obtained by replication.
Now consider pricing the put with strike $K = 100$.

The expected payoff from the put at maturity is

$$(0.60)(0) + (0.40)(10) = 4.$$ 

Discounting this back to the present at the risk-free rate, we obtain

$$\frac{4}{1.02} = 3.92,$$

which is the same price obtained via replication.

As these examples show, risk-neutral pricing is computationally much simpler than replication.
The Black–Scholes model is unambiguously the best known model of option pricing.

Also one of the most widely used: it is the benchmark model for

- Equities.
- Stock indices.
- Currencies.
- Futures.

Moreover, it forms the basis of the Black model that is commonly used to price some interest-rate derivatives such as caps and floors.
The Black-Scholes Model (Cont’d)

- Technically, the Black-Scholes model is much more complex than discrete models like the binomial.
  - In particular, it assumes continuous evolution of uncertainty.
  - Pricing options in this framework requires the use of very sophisticated mathematics.

- What is gained by all this sophistication?
  - Option prices in the Black-Scholes model can be expressed in closed-form, i.e., as particular explicit functions of the parameters.
  - This makes computing option prices and option sensitivities very easy.
Assumptions of the Model

- The main assumption of the Black-Scholes model pertains to the evolution of the stock price.
- This price is taken to evolve according to a geometric Brownian motion.
- Shorn of technical details, this says essentially that two conditions must be met:
  - Returns on the stock have a lognormal distribution with constant volatility.
  - Stock prices cannot jump (the market cannot “gap”).
The Log-Normal Assumption

- The log-normal assumption says that the (natural) log of returns is **normally** distributed: if $S$ denotes the current price, and $S_t$ the price $t$ years from the present, then

  $$\ln \left( \frac{S_t}{S} \right) \sim N(\mu t, \sigma^2 t).$$

- Mathematically, **log-returns** and **continuously-compounded returns** represent the same thing:

  $$\ln \left( \frac{S_t}{S} \right) = x \iff \frac{S_t}{S} = e^x \iff S_t = Se^x.$$

- Thus, log-normality says that returns on the stock, expressed in continuously-compounded terms, are normally distributed.
The number $\sigma$ is called the volatility of the stock. Thus, volatility in the Black-Scholes model refers to the standard deviation of annual log-returns.

The Black-Scholes model takes this volatility to be a constant. In principle, this volatility can be estimated in two ways:

- From historical data. (This is called historical volatility.)
- From options prices. (This is called implied volatility.)

We discuss the issue of volatility estimation in the last part of this segment. For now, assume the level of volatility is known.
Is GBM a Reasonable Assumption?

- The log-normality and no-jumps conditions appear unreasonably restrictive:
  - Volatility of markets is typically not constant over time.
  - Market prices do sometimes “jump.”
  - In particular, the no-jumps assumption appears to rule out dividends.

- Dividends (and similar predictable jumps) are actually easily handled by the model.

- The other issues are more problematic, and not easily resolved. We discuss them in the last part of this presentation.
Option Prices in the Black-Scholes Model

Option prices in the Black–Scholes model may be recovered using a replicating portfolio argument.

Of course, the construction—and maintenance—of a replicating portfolio is significantly more technically complex here than in the binomial model.

We focus here instead on the final option prices that result and the intuitive content of these prices.
Notation

- \( t \): current time.
- \( T \): Horizon of the model. (So time-left-to-maturity: \( T - t \).)
- \( K \): strike price of option.
- \( S_t \): current price of stock.
- \( S_T \): stock price at \( T \).
- \( \mu, \sigma \): Expected return and volatility of stock (annualized).
- \( r \): risk-free rate of interest.
- \( C, P \): Prices of call and put (European only).
The call-pricing formula in the Black-Scholes model is

\[ C = S_t \cdot N(d_1) - e^{-r(T-t)} K \cdot N(d_2) \]

where \( N(\cdot) \) is the *cumulative* standard normal distribution [\( N(x) \) is the probability under a standard normal distribution of an observation less than or equal to \( x \)], and

\[ d_1 = \frac{1}{\sigma \sqrt{T-t}} \left[ \ln \left( \frac{S_t}{K} \right) + (r + \frac{1}{2} \sigma^2)(T-t) \right] \]

\[ d_2 = d_1 - \sigma \sqrt{T-t} \]

This formula has a surprisingly simple interpretation.
To replicate a call in general, we must
- Take a long position in $\Delta_c$ units of the underlying, and
- Borrow $B_c$ at the risk-free rate.

The cost of this replicating portfolio—which is the call price—is

$$C = S_t \cdot \Delta_c - B_c.$$  \hspace{1cm} (1)

The Black-Scholes formula has an identical structure: it too is of the form

$$C = S_t \times [\text{Term 1}] - [\text{Term 2}].$$ \hspace{1cm} (2)
Comparing these structures suggests that

\[ \Delta_c = N(d_1). \]  
\[ B_c = e^{-r(T-t)}K \cdot N(d_2). \]

This is exactly correct! The Black-Scholes formula is obtained precisely by showing that the composition of the replicating portfolio is (3)–(4) and substituting this into (1).

In particular, \( N(d_1) \) is just the delta of the call option.
Recall that to replicate a put in general, we must

- Take a *short* position in $|\Delta_p|$ units of the underlying, and
- Lend $B_p$ at the risk-free rate.

Thus, in general, we can write the price of the put as

$$P = B_p + S_t, \Delta_p$$

The Black-Scholes formula identifies the exact composition of $\Delta_p$ and $B_p$:

$$\Delta_p = -N(-d_1) \quad B_p = PV(K) N(-d_2)$$

where $N(\cdot)$, $d_1$, and $d_2$ are all as defined above.
Therefore, the price of the put is given by

\[ P = PV(K) \cdot N(-d_2) - S_t \cdot N(-d_1). \]  \hspace{1cm} (6)

Expression (6) is the Black-Scholes formula for a European put option.

Equivalently, since \( N(x) + N(-x) = 1 \) for any \( x \), we can also write

\[ P = S_t \cdot [N(d_1) - 1] + PV(K) \cdot [1 - N(d_2)] \]  \hspace{1cm} (7)
Alternative way to derive Black–Scholes formulae: use risk-neutral (or risk-adjusted) probabilities.

To identify option prices in this approach: take expectation of terminal payoffs under the risk-neutral probability measure and discount at the risk-free rate.

The terminal payoffs of a call option with strike $K$ are given by

$$\max\{S_T - K, 0\}.$$ 

Therefore, the arbitrage-free price of the call option is given by

$$C = e^{-rT} E_t^{*} [\max\{S_T - K, 0\}]$$

where $E_t^{*} [\cdot]$ denotes time–$t$ expectations under the risk-neutral measure.
Equivalently, this expression may be written as

\[ C = e^{-rT} E_t^* [S_T - K \mid S_T \geq K] \]

Splitting up the expectation, we have

\[ C = e^{-rT} E_t^* [S_T \mid S_T \geq K] - e^{-rT} E_t^* [K \mid S_T \geq K]. \]

From this to the Black–Scholes formula is simply a matter of grinding through the expectations, which are tedious, but not otherwise difficult.
Specifically, it can be shown that

\[ e^{-rT} E_t^* [S_T \mid S_T \geq K] = S_t \, N(d_1) \]

\[ e^{-rT} E_t^* [K \mid S_T \geq K] = e^{-rT} K \, N(d_2) \]

In particular, \( N(d_2) \) is the risk-neutral probability that the option finishes in-the-money (i.e., that \( S_T \geq K \)).
Remarks on the Black-Scholes Formulae

Two remarkable features of the Black–Scholes formulae:

- Option prices only depend on *five* variables: $S$, $K$, $r$, $T$, and $\sigma$.
- Of these five variables, only *one*—the volatility $\sigma$—is not directly observable.

This makes the model easy to implement in practice.
It must also be stressed that these are *arbitrage-free* prices.

That is, they are based on construction of replicating portfolios that perfectly mimic option payoffs at maturity. Thus, if prices differ from these predicted levels, the replicating portfolios can be used to create riskless profits.

The formulae can also be used to delta-hedge option positions.

For example, suppose we have written a call option whose current delta, using the Black-Scholes formula, is $N(d_1)$. To hedge this position, we take a long position in $N(d_1)$ units of the underlying.

Of course, dynamic hedging is required.
Finally, it must be stressed again that closed-form expressions of this sort for option prices is a rare occurrence.

The particular advantage of having closed forms is that they make computation of option sensitivities (or option “greeks”) a simple task.

Nonetheless, such closed-form expressions exist in the Black-Scholes framework only for European-style options.

For example, closed-forms do not exist for American put options.

However, it is possible to obtain closed-form solutions for certain classes of exotic options (such as compound options or barrier options).
Closed-forms make it easy to compute option prices in the Black-Scholes model:

1. Input values for $S_t$, $K$, $r$, $T - t$, and $\sigma$.
2. Compute $d_1 = \left[ \ln\left(\frac{S_t}{K}\right) + \left( r + \frac{\sigma^2}{2} \right) (T - t) \right] / [\sigma \sqrt{T - t}]$.
3. Compute $d_2 = d_1 - \sigma \sqrt{T - t}$.
4. Compute $N(d_1)$.
5. Compute $N(d_2)$.
6. Compute option prices.

$$C = S_t N(d_1) - e^{-r(T-t)} K N(d_2)$$

$$P = e^{-r(T-t)} K [1 - N(d_2)] - S_t [1 - N(d_1)]$$
The following figure illustrates this procedure.

Four parameters are held fixed in the figure:

\[ K = 100, \quad T - t = 0.50, \quad \sigma = 0.20, \quad r = 0.05 \]

The figures plot call and put prices as \( S \) varies from 72 to 128.

Observe non-linear reaction of option prices to changes in stock price.

- For deep OTM options, slope \( \approx 0 \).
- For deep ITM options, slope \( \approx 1 \).

Of course, this slope is precisely the option delta!
Plotting Option Prices

Option Pricing: A Review
Rangarajan K. Sundaram

Introduction
Pricing
Options by Replication
The Option Delta
Option Pricing using Risk-Neutral Probabilities
The Black-Scholes Model
Implied Volatility

Plotting Option Prices

Option Values

Stock Prices

Call
Put
Implied Volatility

- Given an option price, one can ask the question: what level of volatility is implied by the observed price?
- This level is the *implied volatility*.
- Formally, implied volatility is the volatility level that would make observed option prices consistent with the Black-Scholes formula, given values for the other parameters.
For example, suppose we are looking at a call on a non-dividend-paying stock.

Let $K$ and $T - t$ denote the call’s strike and time-to-maturity, and let $\hat{C}$ be the call’s price.

Let $S_t$ be the stock price and $r$ the interest rate.

Then, the implied volatility is the unique level $\sigma$ for which

$$C^{bs}(S, K, T - t, r, \sigma) = \hat{C},$$

where $C^{bs}$ is the Black-Scholes call option pricing formula.

Note that implied volatility is uniquely defined since $C^{bs}$ is strictly increasing in $\sigma$. 
Implied volatility represents the “market’s” perception of volatility anticipated over the option’s lifetime.

Implied volatility is thus *forward looking*.

In contrast, historical volatility is *backward looking*. 
In theory, any option (any $K$ or $T$) may be used for measuring implied volatility.

Thus, if we fix maturity and plot implied volatilities against strike prices, the plot should be a flat line.

In practice, in equity markets, implied volatilities for “low” strikes (corresponding to out-of-the-money puts) are typically much higher than implied volatilities for ATM options. This is the volatility skew.

In currency markets (and for many individual equities), the picture is more symmetric with way-from-the-money options having higher implied volatilities than at-the-money options. This is the volatility smile.

See the screenshots on the next 4 slides.
Introduction to Option Pricing

Pricing Options by Replication

The Option Delta

Option Pricing using Risk-Neutral Probabilities

The Black-Scholes Model

Implied Volatility

S&P 500 Implied Volatility Plot

JUN Volatility Skew
Last Trading Date: June 17, 2004

S&P 500 (SP) Option Volatility Analysis - Mozilla Firefox

Getting Started  Latest Headlines

Implied Volatility Plot
S&P 500 Implied Volatility Plot

SEP Volatility Skew
Last Trading Date: September 17, 2004

SEP Fut=1141.10 Days=118 atmVol=16.22% IntRate = 6.50%
USD-GBP Implied Volatility Plot

Introduction

Pricing Options by Replication

The Option Delta

Option Pricing using Risk-Neutral Probabilities

The Black-Scholes Model

Implied Volatility

<table>
<thead>
<tr>
<th>Strike</th>
<th>CALL 1</th>
<th>CALL 2</th>
<th>CALL 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>165</td>
<td>15.6</td>
<td>15.2</td>
<td>14.9</td>
</tr>
<tr>
<td>166</td>
<td>15.6</td>
<td>15.2</td>
<td>14.9</td>
</tr>
<tr>
<td>167</td>
<td>15.6</td>
<td>15.2</td>
<td>14.9</td>
</tr>
<tr>
<td>168</td>
<td>15.6</td>
<td>15.2</td>
<td>14.9</td>
</tr>
<tr>
<td>169</td>
<td>15.6</td>
<td>15.2</td>
<td>14.9</td>
</tr>
<tr>
<td>170</td>
<td>15.6</td>
<td>15.2</td>
<td>14.9</td>
</tr>
<tr>
<td>171</td>
<td>15.6</td>
<td>15.2</td>
<td>14.9</td>
</tr>
<tr>
<td>172</td>
<td>15.6</td>
<td>15.2</td>
<td>14.9</td>
</tr>
<tr>
<td>173</td>
<td>15.6</td>
<td>15.2</td>
<td>14.9</td>
</tr>
<tr>
<td>174</td>
<td>15.6</td>
<td>15.2</td>
<td>14.9</td>
</tr>
<tr>
<td>175</td>
<td>15.6</td>
<td>15.2</td>
<td>14.9</td>
</tr>
</tbody>
</table>

MAY Volatility Skew

Last Trading Date: May 7, 2004

www.ampublishing.com
USD-GBP Implied Volatility Plot

We are currently in the process of adding features to the site. Please do not hesitate to let us know what trading tools you would like to have. You can submit your email address if you want a reply or if you want to be informed of future updates.

Suggestions: 

E-mail: 

Submit
Two sources are normally ascribed for the volatility skew.

One is the returns distribution. The Black-Scholes model assumes log-returns are normally distributed.

However, in every financial market, extreme observations are far more likely than predicted by the log-normal distribution.

- Extreme observations = observations in the tail of the distribution.
- Empirical distributions exhibit “fat tails” or leptokurtosis.

Empirical log-returns distributions are often also skewed.
Source of the Volatility Skew (Cont’d)

- Fat tails $\implies$ Black-Scholes model with a constant volatility will *underprice* out-of-the-money puts relative to those at-the-money.

- Put differently, “correctly” priced out-of-the-money puts will reflect a higher implied volatility than at-themoney options.

- This is exactly the volatility skew! That is, the volatility skew is evidence not only that the lognormal model is not a fully accurate description of reality but also that the market recognizes this shortcoming.

- As such, there is valuable information in the smile/skew concerning the “actual” (more accurately, the market’s expectation) of the return distribution. For instance:
  - More symmetric smile $\implies$ Less skewed distribution.
  - Flatter smile/skew $\implies$ Smaller kurtosis.
The other reason commonly given for the volatility skew is that the world as a whole is net long equities, and so there is a positive net demand for protection in the form of puts.

This demand for protection, coupled with market frictions, raises the price of out-of-the-money puts relative to those at-the-money, and results in the volatility skew.

With currencies, on the other hand, there is greater symmetry since the world is net long both currencies, so there is two-sided demand for protection.

Implicit in this argument is Rubinstein’s notion of “crash-o-phobia,” fear of a sudden large downward jump in prices.
Obvious question: why not generalize the log-normal distribution?

Indeed, there may even be a “natural” generalization.

The Black-Scholes model makes two uncomfortable assumptions:

- No jumps.
- Constant volatility.

If jumps are added to the log-normal model or if volatility is allowed to be stochastic, the model will exhibit fat tails and even skewness.
Generalizing Black-Scholes (Cont’d)

- So why don’t we just do this?
  - We can, but complexity increases (how many new parameters do we need to estimate?).
  - Jumps & SV have very different dynamic implications.
  - Reality is more complex than either model (indeed, substantially so).

- Ultimately: better to use a simple model with known shortcomings?

- On jumps vs stochastic volatility models, see