Merton-model Approach to Valuing Correlation Products

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Credit Risk Elective
Spring 2009
Binomial with Merton Model

• Important method for calculating distribution of loan losses:
  ✓ widely used in banking
  ✓ used by Basel II regulations to set bank capital requirements

• Linked to distance-to-default analysis
Mixed Binomial: Using Merton’s Models as Mixing Distribution

• In Merton model value of risky debt depends on firm value and default risk is correlated because firm values are correlated (e.g., via common dependence on market factor).

• Value of firm $i$ at time $T$:

$$V_{T,i} = V_i \exp\left( \left( \mu_i - (1/2)\sigma_d^2 \right)V_i \right) + \sigma_d \sqrt{T} \epsilon_i$$

expected value of $R_C$
surprise in $R_C$

where $\epsilon_i \sim N(0, 1)$

• We will assume that correlation between firm values arises because of correlation between surprise in individual firm value ($\epsilon_i$) and market factor ($m$)
Mixed Binomial: Using Merton’s Models as Mixing Distribution

• Suppose correlation between each firm’s value and the market factor is the same and equal to \( \sqrt{\rho} \).
• This means that we may model correlation between the \( \varepsilon \)’s as

\[
\varepsilon_i = \sqrt{\rho} m + \sqrt{1-\rho} v_i, \quad i = 1, \ldots, N
\]

and

\[
\text{corr}(\varepsilon_i, \varepsilon_j) = \rho
\]

• Where \( m \) and \( v_i \) are independent \( N(0,1) \) random variables and \( \rho \) is common to all firms
• Notice that if \( v_i \sim N(0,1) \) and \( m \sim N(0,1) \) then \( \varepsilon_i \sim N(0,1) \)
Structural Approach, contd.

• From our analysis of *distance-to-default*, we know that under the Merton Model a firm defaults when:

$$
\varepsilon_i - R_{Di} = (\mu_i - \frac{1}{2}\sigma_{V,i}^2)T / \sigma_{V,i} \sqrt{T} \text{ where } R_{Di} = \ln(B_i / V_i)
$$

• The *unconditional* probability of default, $p$, is therefore:

$$
p < \text{Prob} \left( \varepsilon_i \left( \frac{R_{Di} - (\mu_i - \frac{1}{2}\sigma_{V,i}^2)}{\sigma_{V,i} \sqrt{T}} \right) = N \left( \frac{R_D - (\mu_i - \frac{1}{2}\sigma_{V,i}^2)}{\sigma_i \sqrt{T}} \right) \right)
$$

• In this model we assume that the *default probability*, $p$, is *constant across firms*
Structural Approach to Correlation – the Idea

• Working out the distribution of portfolio losses directly when the $\varepsilon$’s are correlated is not easy
• But, if we work out the *distribution conditional on the market shock*, $m$, then we can exploit the fact that the remaining shocks are independent and work out the portfolio loss distribution
Structural Approach, contd.

• The shock to the return, $\varepsilon_i$, is related to the common and idiosyncratic shocks by:

$$\varepsilon_i = \sqrt{\rho_m} + \sqrt{1 - \rho} \nu_i$$

• Default occurs when:

$$\varepsilon_i = \sqrt{\rho_m} + \sqrt{1 - \rho} \nu_i < \frac{R_{D,i} - (\mu_i - \frac{1}{T} \sigma_{V,i}^2)}{\sigma_{V,i} \sqrt{T}} = N^{-1}(p)$$

or

$$\nu_i < \frac{N^{-1}(p) - \sqrt{\rho_m}}{\sqrt{1 - \rho}}$$
The Default Condition

\[ \nu_i < \frac{N^{-1}(p) - \sqrt{\rho}m}{\sqrt{1 - \rho}} \]

- A large value of \( m \) means a “good” shock to the market (high asset values)
- The larger the value of \( m \) the more negative the idiosyncratic shock, \( \nu_i \), has to be to trigger default
- The higher the correlation, \( \rho \), between the firm shocks, the larger the impact of \( m \) on the critical value of \( \nu_i \).
Structural Approach, contd.

• Conditional on the realisation of the common shock, $m$, the probability of default is therefore:

$$\text{Prob}(\text{default} | m) = \text{Prob} \ v_i < \frac{N^{-1}(p) - \sqrt{\rho m}}{\sqrt{1-\rho}}$$

$$= N \ \frac{N^{-1}(p) - \sqrt{\rho m}}{\sqrt{1-\rho}} = \theta(m), \text{ say}$$

and therefore $\frac{N^{-1}(p) - \sqrt{\rho m}}{\sqrt{1-\rho}} = N^{-1}(\theta)$
The relation between $m$ and $\theta$

- For a given market shock, $m$, $\theta$ gives the conditional probability of default on an individual loan
Implications of Conditional Independence

• For a given value of \( m \), as the number of loans in the portfolio \( \to \infty \), the *proportion* of loans in the portfolio that default converges to the probability \( \theta \)

• The probability that the loan-loss proportion, \( L \), is \( < \theta \) is therefore:

\[
\text{Prob}(L = \theta) = \text{Prob} \left( \frac{N^{-1}(p) - \sqrt{\rho m}}{\sqrt{1 - \rho}} \right) = \text{Prob} \left( m \right) \frac{1}{\sqrt{\rho}} \left( N^{-1}(p) - N^{-1}(\theta) \sqrt{1 - \rho} \right)
\]

\[
= \text{Prob} \left( m - \frac{1}{\sqrt{\rho}} \left( \sqrt{1 - \rho} \ N^{-1}(\theta) - N^{-1}(p) \right) \right)
\]

\[
= N \frac{1}{\sqrt{\rho}} \left( \sqrt{1 - \rho} \ N^{-1}(\theta) - N^{-1}(p) \right)
\]
Loan Loss Distribution – Structural Model

\[ \text{Prob}(L = \theta) = N \left( \frac{1}{\sqrt{\rho}} \left( \sqrt{1 - \rho} \left( N^{-1}(\theta) - N^{-1}(p) \right) \right) \right) \]

- This result gives the distribution of the fraction of loans that default in a well diversified homogeneous portfolio where the correlation in default comes from dependence on a common factor.

- **Homogeneity** means that each loan has:
  - the same default probability, \( p \)
  - (implicitly) the same loss-given-default
  - the same correlation, \( \rho \), across different loans

- The distribution has **two parameters**
  - default probability, \( p \)
  - correlation, \( \rho \)
Loan Loss Distribution with $p = 1\%$ and $\rho = 12\%$ and $0.6\%$

Portfolio Loan Loss (%)

- $p = 1.5\%$  $\rho = 12.0\%$
- $p = 1.5\%$  $\rho = 0.6\%$
Example of Vasicek formula Applied to Bank Portfolio

Source: Vasicek