The Copula Approach to Valuing Correlation Products

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Modelling Correlation:

- **Default** is a *binomial* event: it happens or it doesn’t
- With a fixed recovery rate the distribution of portfolio losses is the distribution of the *number of defaults*
- But *difficult* to include default *correlation* directly into standard binomial framework
- Two common approaches:
  - Copula Approach
    - widely used in pricing – but needs caution
  - Structural Approach: sounder approach... in future?
Caution over Copulas

• The copula approach means is that we can always separate the dependence structure between two or more random variables from their unconditional (or marginal) distributions.

• Sounds very powerful BUT problem is that often very little guidance available in how to choose copula

• Gaussian copula (method described here) widely used in practice but quite possibly a poor description of reality.
Simulating Default Times

• The starting point is the intensity model with constant intensity $\lambda$
• Under this model the probability of survival up to time $t$ is:

$$p(\tau > t) = \exp(-\lambda t)$$

• As with any cumulative distribution, if we were to make a random drawing from the distribution of default times, $\tau$, the cumulative probability $p(\tau > t)$ would be equally likely to be anywhere within the range zero to one (see further intuition below).
Simulating default time in Intensity Model

$U$ drawn from uniform distribution on $[0,1]$

$\text{default time}$

Survival Probability

Survival time
Inverse Cumulative Method for Random Numbers: Intuition

*See diagram on next but one slide*

- We wish to make a random drawing from the distribution of default times.
- The diagram shows both the probability density and the cumulative distribution.
- The total area under the density is *one*: suppose we divide up this area into 10 equal regions (marked by the vertical dotted lines).
- A default time drawn at random would be equally likely to fall into any of these 10 intervals.
- We now use the following rule:
  - randomly draw a number between 1 and 10
  - use this to choose one of the 10 intervals
  - our random number is the value of the default time in the middle (say) of the interval.

*continued next slide*
Inverse Cumulative Method: Contd.

• All that is required to implement this method is to know where the boundaries of the intervals lie.

• With 10 intervals each interval accounts for 10% of the probability and so the cumulative probability at the first boundary is 10%, at the second it is 20% and so forth.

• *We can simply look up these values on the cumulative distribution:* notice that the default time boundaries for the probability density (horizontal axis) correspond to the 10%, 20% etc. points on the cumulative distribution.

• We could therefore implement the method as follows:
  ✓ Choose a number \((k)\) from one to 1 to 10
  ✓ Look up the value of the default times that corresponds to cumulative probabilities of \((k -1)* 10\% \) and \(k*10\%\) (the left and right hand boundaries for the \(k^{th}\) interval) and choose the number in the middle.

• The actual method we use (choosing \(U\) from a uniform distribution on \([0,1]\)) is equivalent to doing this with an infinite number of intervals.
Inverse Cumulative Method for Random Numbers: Intuition
Simulating Default Times

- In summary, therefore, to simulate a default time $\tau$ in the intensity model we:
  1. choose a random number, $U$, so that it is equally likely to be anywhere in the range $\{0,1\}$ – i.e., from a uniform distribution on $[0,1]$.
  2. solve:

\[
U = \exp(-\lambda \tau) \quad \tau = -\frac{1}{\lambda} \ln(U)
\]

and the value $\tau$ of we obtain is a random drawing from the distribution of default times.
The Gaussian Copula Method for Default-Time Correlation and FTD Valuation

• To simulate *correlated default times* for FTD and CDO valuation an approach known as the *Gaussian Copula Method* is often used

• Correlation is modelled either through dependence on a *single common factor* or (sometimes) from a general correlation matrix

• Using the single common factor approach: if the correlation between each *pair of names* is \( \rho \) then for \( N \) names we calculate correlated random variables \( \varepsilon_1, \ldots, \varepsilon_N \) as:

\[
\varepsilon_i = \sqrt{\rho} m + \left( \sqrt{1-\rho} \right) v_i, \quad i = 1, \hat{W}, N, \quad m \sim N(0,1) \text{ and } v_i \sim N(0,1)
\]

Note: \( m \) and \( v_i \) and \( v_i \) and \( v_j \) are independent and therefore

\[
\text{corr}(\varepsilon_i, \varepsilon_j) = \text{cov}(\varepsilon_i, \varepsilon_j) = \rho
\]
Generating Correlated Default Times

• For each trial in the simulation:
  ✓ generate N correlated values of $\varepsilon$ (as on previous slide) – one for each name/credit
  ✓ for each of the $\varepsilon$’s, calculate the corresponding default time as:

  $$\tau_i = -\frac{1}{\lambda} \ln(U_i) \quad \text{where} \quad U_i = N(\varepsilon_i)$$

  and $N(.)$ represents the cumulative normal distribution

• For an FTD, calculate the minimum time-to-default and, if this is less than the contract maturity record a default
Generating Correlated Default Times

- Simulated correlated $U_1$, $U_2$ etc. .. And use these to generate correlated default times
Valuing an FTD – The Basic Idea

• Using simulation
  1. value loss leg up to time of default or end of contract, which ever comes first
  2. value premium leg for 1 b.p. – again, up to time of default or end of contract, which ever comes first
  3. find premium that equates value of loss and premium legs
Valuing an FTD

• Value of the *loss leg* of the FTD
  ✓ expected discounted value of the loss leg

• Value of the *premium leg* (for a 1 b.p. fee, for example)
  ✓ Expected discounted value of the 1 b.p. fee stream to default or maturity, whichever is shorter

• Dividing the value of the loss leg by the value of a 1 b.p. per year premium leg,
• we obtain the FTD premium.

• As already noted: $\sqrt{\rho}$ is *correlation* between each *firm* and *market* and so *correlation* between each *pair of firms* is $\rho$.
Next week ... CDOs, tranches etc.