# Credit Risk: Simple Closed Form Approximate Maximum Likelihood Estimator, Large deviations and fast simulation

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# The credit risk problems considered

- We want to estimate the conditional default probability of any firm as a function of given global and company specific information.
- This over short time periods - a month or a quarter, as well as longer time horizons.
- Analogous to predicting a person's health (mortality) as a function of his blood pressure, sugar, cholesterol, pollution, income, taxes (un)paid, etc.
- We model in discrete time and assume conditional probabilities have a popular default intensity type or logit type form

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# Big picture - contributions

- The literature posits a parametric form for conditional default probabilities. Solves for parameters by maximising the likelihood function.
- Computationally intensive, solution has a black box flavour - drivers of the parameters not clear.
- We observe, in some popular settings, that since these probabilities are small, and co-variates can be transformed to be Gaussian, the MLE has a simple closed form approximation

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- These are almost as good as MLE when the model is correctly specified - Performance slightly worsens for large number of firms (5,000 plus), large default probabilities (5%)

- Equally good or equally bad for mis-specified models, including on empirical data.

- We characterize the performance of the proposed approximate MLE as well as MLE in an asymptotic regime - probabilities decrease to zero, number of firms and number of time periods increase to infinity

- Some numerical/empirical support

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# The credit risk problems maybe not considered ...

- We analyze the portfolio credit risk problem and develop an asymptotic regime where
  - we conduct large deviations tail analysis of large losses
  - Develop fast simulation techniques for computing tail risk measures
- Both in calibration as well as in portfolio analysis, we include contagion effects in the model in a meaningful way.

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# Review and models considered

# Popular reduced form intensity based models

- For each firm $i$, default explaining covariates such as prevailing interest rates, GDP, distance to default, cash over total assets, modelled as a continuous Markov process, example,

$$dX_{i,t} = A_i(\theta_i - X_{i,t})dt + \Sigma_i dW_{i,t}$$

for $0 \leq t \leq T$.

- Firm $i$ has a doubly stochastic default intensity process

$$\lambda_i(t) = \Lambda_i(\beta, X_{i,t})$$

where $\beta$ is the set of parameters to be estimated.

- **Conditioned on the covariates, default is an arrival from a non-homogeneous Poisson process.**

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# Conditional probability under discretization

$$P(\text{ no default by } t+1 | \text{ no default by time } t) = E_t e^{-\int_t^{t+1} \lambda_i(s) ds}$$

- If we assume that over time period $[t, t+1)$

$$\lambda_i(s) = \exp(\beta^\mathsf{T} X_{i,t} - \alpha)$$

then

$$P(\text{ default by } t+1 | \text{ no default by time } t) = 1 - \exp(-\exp(\beta^\mathsf{T} X_{i,t} - \alpha))$$

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# Discrete Logit models

- Covariates affecting Firm $i$ follow a stationary process $\{X_{i,t}\}$
- Conditional default probability at period $t$ to default in $[t, t+1)$

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# Model and Maximum likelihood estimation

# We consider a discrete time model

- Covariates - Autoregressive process

$$X_{i,t+1} = \mathbf{A}X_{i,t} + \tilde{\mathbf{E}}_{i,t+1}$$

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$$p(X_{i,t}) = 1 - \exp\left(-\exp(\beta^T X_{i,t} - \alpha)\right)$$

or

$$p(X_{i,t}) = \frac{\exp(\beta^T X_{i,t} - \alpha)}{1 + \exp(\beta^T X_{i,t} - \alpha)},$$

- Parameters $\beta$, $\alpha$ need to be estimated from data. Duffie et al 2006, Duan et al 2007, Chava and Jarrow 2004, Duan, Sun Wang 2012, Shumway 2002.

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# Maximum likelihood method to estimate parameters

- Default data

$$(x_{i,t}, d_{i,t}) \text{ for } t = 1, , 2, \ldots, T, i = 1, \ldots, m,$$

where $d_{i,t} = 1$ if company $i$ defaults in $[t, t+1)$ and zero otherwise.

- Likelihood $\mathcal{L}$ of seeing the data

$$\mathcal{L} = \prod_{i,t} p(x_{i,t})^{d_{i,t}}(1 - p(x_{i,t}))^{1-d_{i,t}}$$

- This is optimized numerically to find $\beta$ and $\alpha$.
- Computationally intensive; black box.

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# MLE: Default intensity model

- $$p(x_{i,t}) = 1 - \exp(-e^{\beta^\intercal x_{i,t} - \alpha}).$$

- Setting the partial derivatives w.r.t. $(\beta, \alpha)$ to zero,

$$\sum_{i,t} \frac{x_{i,t} e^{\beta^\intercal x_{i,t} - \alpha}}{1 - \exp(-e^{\beta^\intercal x_{i,t} - \alpha})} d_{i,t} = \sum_{i,t} x_{i,t} e^{\beta^\intercal x_{i,t} - \alpha}$$

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$$\sum_{i,t} \frac{e^{\beta^\intercal x_{i,t} - \alpha}}{1 - \exp(-e^{\beta^\intercal x_{i,t} - \alpha})} d_{i,t} = \sum_{i,t} e^{\beta^\intercal x_{i,t} - \alpha}.$$

# MLE: Logit model

- $$p(x_{i,t}) = \frac{\exp(\beta^\intercal x_{i,t} - \alpha)}{1 + \exp(\beta^\intercal x_{i,t} - \alpha)}.$$

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# Key insight

- Re-express

$$\frac{1}{\#\ i,t}\sum_{i,t} d_{i,t} = E\left(\exp(\beta^{\mathsf{T}} X_{i,t} - \alpha)\right)$$

$$+\left(\frac{1}{\#\ i,t}\sum_{i,t}\frac{\exp(\beta^{\mathsf{T}} x_{i,t} - \alpha)}{1+\exp(\beta^{\mathsf{T}} x_{i,t} - \alpha)} - E\left(\exp(\beta^{\mathsf{T}} X_{i,t} - \alpha)\right)\right).$$

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- When $X$ is Gaussian:

$$E\left[\exp(\beta^{\mathsf{T}}X)\right] = \exp(\frac{1}{2}\beta^{\mathsf{T}}\Sigma\beta) \text{ and } E\left[X\exp(\beta^{\mathsf{T}}X)\right] = \Sigma\beta\exp(\frac{1}{2}\beta^{\mathsf{T}}\Sigma\beta).$$

- This suggests that we set the estimator to dominant term

$$\hat{\beta} = \Sigma_i^{-1}\left(\frac{\sum_{i,t} x_{i,t} d_{i,t}}{\sum_{i,t} d_{i,t}}\right).$$

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# How good is the estimator

# Analysis of estimator quality in asymptotic regime

- We model conditional default probabilities as

$$p_\gamma(X_{i,t}) = \exp(\beta^T X_{i,t} - \alpha(\gamma))(1 + o_\gamma(1))$$

where $\alpha(\gamma)$ is of order $\log(1/\gamma)$, $\{X_t\}$ is a vector autoregressive process.

- Conditional probabilities of order $\gamma$ ( $\approx 10^{-3}$)
- Number of companies is of order $\frac{1}{\gamma^\delta}$ for $\delta > 0$
- Number of time periods of provided data is $\frac{1}{\gamma^\zeta}$ for $\zeta \in (0, 1)$.

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- Our estimator in this regime

$$\hat{\beta} = \Sigma_{X,X}^{-1} \left( \frac{\sum_{i,t} X_{i,t} D_{i,t}}{\sum_{i,t} D_{i,t}} \right)$$

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- Recall that $\{X_t\}$ captures the underlying covariates. $D_{i,t}$ are default indicators.
- This converges to $\beta$ as $\gamma \to 0$.

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# Mean square error analysis: Proposed estimator

- Theorem: The mean square error

$$||\hat{\beta} - \beta||^2 = \Theta(\gamma^{\delta+\zeta-1}) + \Theta(\gamma^{\zeta}).$$

- $\delta + \zeta < 1$: No defaults asymptotically
- $\delta < 1$: Increasing $\delta$ helps the estimator. *More firms in dataset improve the estimator*
- $\delta > 1$: increasing $\delta$ no longer matters. *Insensitive to increase in the number of firms*

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# Model misspecification: An illustration

- Model generating defaults has two Gaussian factors common to all firms:

$$\frac{\exp(\beta_1 Y_{1,t} + \beta_2 Y_{2,t} - \alpha(\gamma))}{1 + \exp(\beta_1 Y_{1,t} + \beta_2 Y_{2,t} - \alpha(\gamma))},$$

  $Y_{1,t}$ and $Y_{2,t}$ are assumed to have zero mean, variance 1 and correlation $\rho$

- Only the first factor with time series $(Y_{1,t} : 1 \leq t \leq T(\gamma))$ is assumed to be relevant by modeller.

- Both estimators asymptotically converge to

$$\hat{\beta}_1 = \beta_1 + \rho\beta_2$$

  and

$$\hat{\alpha} = \alpha(\gamma) - \frac{\beta_2^2(1-\rho^2)}{2}.$$

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# Simulation Experiments

# Comparison of RMSE for default probability 1% per annum, model correctly specified

| Time in months | No. of firms | RMSE($\beta_{prop}$) | RMSE($\beta_{ML}$) |
|---|---|---|---|
| 200 | 1000 | 0.1280 | 0.1248 |
| 200 | 3000 | 0.0787 | 0.0707 |
| 200 | 5000 | 0.0685 | 0.0574 |
| 200 | 7000 | 0.0574 | 0.0435 |
| 200 | 10000 | 0.0547 | 0.0374 |
| 100 | 2000 | 0.1232 | 0.1157 |
| 300 | 2000 | 0.0774 | 0.0714 |
| 500 | 2000 | 0.0608 | 0.0547 |
| 700 | 2000 | 0.0565 | 0.0509 |

True Parameters: $(\alpha = 7.5, \beta_1 = -0.2, \beta_2 = 0.5, \beta_3 = 0.5)$. RMSE of the proposed estimator is only slightly larger than that of MLE except when the no. of companies is large.

# Comparison of RMSE for default probability 1% per annum, missing covariate with small and large coefficient

| $\beta_3$ | No. of firms | RMSE($\beta_{prop}$) | RMSE($\beta_{ML}$) |
|---|---|---|---|
| 0.5 | 1000 | 0.1403 | 0.1392 |
| 0.5 | 3000 | 0.0871 | 0.0842 |
| 0.5 | 5000 | 0.0741 | 0.0721 |
| 0.5 | 7000 | 0.0754 | 0.0707 |
| 2 | 1000 | 0.3109 | 0.3231 |
| 2 | 3000 | 0.2958 | 0.3041 |
| 2 | 5000 | 0.3046 | 0.3135 |
| 2 | 7000 | 0.3014 | 0.3072 |

True Parameters: ($\alpha = 7.5, \beta_1 = -0.2, \beta_2 = 0.5$), $\beta_3$. Time period 200. Both the proposed estimator and MLE estimate ($\alpha, \beta_1, \beta_2$) only. The RMSE of the two methods is nearly identical. It worsens as value of $\beta_3$ increases.

# Empirical Analysis

# Sample Data Characteristics (From Risk Management Institute, NUS)

1. Number of Companies: 2,000
2. Time Periods:251
3. Defaults:168
4. Default Probability per year: 1.12%
5. Number of Variables Available: 8

# Description of Variables

**Macroeconomic Variables**

1. IndexReturn: trailing 1-year return on the S&P500 index
2. Treasury rate: 3-month US Treasury bill rate

**Firm-Specific Variables**

1. DTD: firms distance-to-default
2. CASH/TA: ratio of the sum of cash and short-term investments to the total assets
3. NI/TA: ratio of net income to the total assets
4. SIZE: log of the ratio of firms market equity value to the average market equity value of the S&P500 firm
5. M/B: market-to-book asset ratio
6. SIGMA: 1-year idiosyncratic volatility

# Assumption of Normality/Boundedness

Figure: Frequency Plots of Variables without Transformation

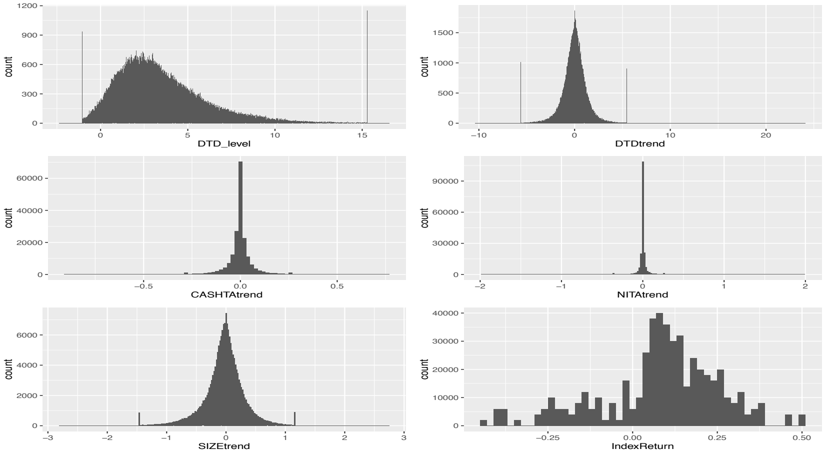

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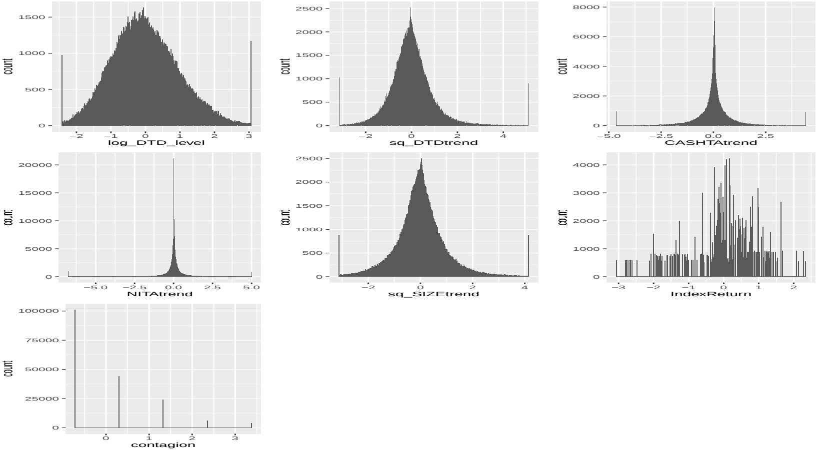
Figure: Frequency Plots of Variables after Transformation



# Calibration Betas

Table: Combined Beta Table

| Decile | Our Calibration | Duffie's MLE | Logit |
|---|---|---|---|
| Constant | -9.251 | -6.739 | -9.344 |
| log_DTD_level | -1.330 | -0.425 | -1.837 |
| sq_DTDtrend | -0.199 | 0.320 | -1.267 |
| CASHTAtrend | -0.035 | 0.006 | -0.045 |
| NITAtrend | -0.417 | -0.108 | -0.060 |
| sq_SIZEtrend | -1.477 | -0.615 | -0.565 |
| IndexReturn | -0.342 | -0.089 | -0.218 |

# Calibration Results

Table: Combined Accuracy Table

| Decile | Our Calibration | Duffie's MLE | Logit |
|---|---|---|---|
| 1 | 0.895 | 0.842 | 0.763 |
| 2 | 0.974 | 0.947 | 0.921 |
| 3 | 0.974 | 0.974 | 0.947 |
| 4 | 1 | 0.974 | 0.947 |
| 5 | 1 | 0.974 | 0.947 |
| 6 | 1 | 0.974 | 0.974 |
| 7 | 1 | 0.974 | 1 |
| 8 | 1 | 1 | 1 |
| 9 | 1 | 1 | 1 |

# Calibration Betas with Contagion

Table: Combined Beta Table with Contagion

| Variable | Our Calibration | Duffie's MLE | Logit |
|---|---|---|---|
| Constant | -9.806 | -6.811 | -9.145 |
| log_DTD_level | -1.281 | -0.322 | -1.587 |
| sq_DTDtrend | -0.174 | 0.072 | -1.235 |
| CASHTAtrend | -0.033 | 0.181 | -0.042 |
| NITAtrend | -0.410 | 0.223 | -0.061 |
| sq_SIZEtrend | -1.462 | -0.755 | -0.582 |
| IndexReturn | 0.021 | 0.007 | -0.198 |
| Contagion | 1.117 | 0.194 | 0.046 |

# Calibration Results with Contagion

Table: Combined Accuracy Table with Contagion

| Decile | Our Calibration | Duffie's MLE | Logit |
|---|---|---|---|
| 1 | 0.921 | 0.868 | 0.763 |
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Table: Computer Generated Data Coefficient Results

| | Underlying Logit Betas | Our Betas | Duffie Betas |
|---|---|---|---|
| Constant | -7.600 | -7.490 | -7.566 |
| CVar1 | 0.225 | 0.198 | 0.224 |
| CVar2 | 0.549 | 0.536 | 0.561 |
| CVar3 | -1.417 | -1.376 | -1.408 |
| MVar1 | 0.500 | 0.455 | 0.517 |
| MVar2 | 0.700 | 0.687 | 0.664 |

# Portfolio Credit Risk: Tail Analysis

# Our portfolio framework

- Consider a portfolio with $n$ borrowers.

- For each obligor $i$, the conditional default probability in period $[t, t+1)$ has the form

$$p_{i,t} = F(-\alpha_i + \beta^T X_{i,t})$$

where $F$ is a strictly increasing distribution function.

- The covariates follow a vector autoregressive process

$$X_{i,t+1} = \mathbf{A} X_{i,t} + \tilde{\mathbf{E}}_{i,t+1}$$

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- Let $D_{i,t}$ denote the event that obligor $i$ defaults at time $t$. Then, loss suffered equals $e_i$. This may be random.

- One illustrative performance measure of interest may be

$$P(\sum_{i=1}^{n} e_i I(D_{i,t_1}) \geq n a_{t_1}, \sum_{i=1}^{n} e_i I(D_{i,t_2}) \geq n a_{t_2})$$

That is, large losses observed jointly in two time periods $t_1$ and $t_2$.

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# Monte Carlo methodology to compute large loss probabilities

- Start with the value $X_{i,0}$ as well as $p_{i,0}$ for each obligor. Check how many default in period $[0, 1)$.
- Increment the factors generating samples of $\tilde{\mathbf{E}}_1$,

$$X_{i,1} = \mathbf{A}X_{i,0} + \tilde{\mathbf{E}}_{i,1}$$

- Generate samples of $Y_{i,1}$ and compute the conditional probabilities

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# Some related literature

- Dembo, Deuschel, Duffie (2004) - single period, single factor large deviations.
- Glasserman and Li (2005), Glasserman, Kang, Shahabuddin (07, 08). Single period, Gaussian Copula, large deviations, fast simulation.
- Bassamboo, Juneja, Zeevi (2008) T-Copula single period large deviations, fast simulation.
- Giesecke, Spiliopoulos, R. Sowers, and J. Sirignano (2015). Continuous time model, analysis relatively complex.
- Duan, Sun, Wang (2012). Discrete time multi period model. No large deviations analysis.

# Tail Analysis of Large Losses

## Embedding the portfolio credit risk problem in asymptotic regime

- Consider a portfolio with $n$ obligors. We analyze this portfolio as $n \to \infty$.

- For each obligor $i$, the conditional default probability in period $[t, t+1)$ has the form

$$p_{i,t}^{(n)} = F(-m_n\alpha + \tilde{m}_n\beta^T X_t + \tilde{m}_n\eta^T Y_{i,t})$$

- The common factors again follow a vector autoregressive process independent of $n$,

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# Illustrative large deviations result

**Theorem**

*When, $\frac{m_n}{\tilde{m}_n} \to \infty$, and $\liminf_n \tilde{m}_n > 0$, under mild conditions,*

$$\lim_{n\to\infty} \frac{1}{n} \log P\left(\sum_{i=1}^{n} e_i I(D_{i,t}) \geq na\right) = -q(t),$$

*where $q(t)$ equals*

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*Note that it strictly reduces with $t$.*

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# Fast Simulation of Large Losses

# Illustrative rare event simulation problem

- Consider the problem of estimating probability of eighty or more heads in hundred tosses of a fair coin. $(5.58 \times 10^{-10})$.

- Estimator from average of $n$ independent samples

$$\frac{1}{n}\sum_{i=1}^{n} I_i(X_1 + X_2 + \cdots + X_{100} \geq 80).$$

- On average $1.8 \times 10^9$ samples needed to observe a successful sample

- $2.75 \times 10^{12}$ trials needed to get 95% confidence interval of width 5% of the true value.

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# Importance sampling to the rescue

- Generate these samples under a new distribution such that $X_i$'s independently equal 1 with probability $p$.

- Unbias the result using the 'likelihood ratio'

$$\frac{1}{n}\sum_{i=1}^{n} I_i(X_1+\cdots+X_{100}\geq 80)\frac{(1/2)^{\sum_{i=1}^{100}X_i}(1/2)^{100-\sum_{i=1}^{100}X_i}}{p^{\sum_{i=1}^{100}X_i}(1-p)^{100-\sum_{i=1}^{100}X_i}}.$$

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- Consider estimating the rare event probability $P(A)$.

$$P(A) = EI(A) = \int_{x \in A} f(x)dx = \int_{x \in A} \frac{f(x)}{f^*(x)} f^*(x)dx = E^*[LI(A)]$$

where $L(x) = \frac{f(x)}{f^*(x)}$ is called the likelihood ratio.

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- Challenge is to find $f^*$ that minimizes the variance or the second moment of the estimator $L * I(A)$.

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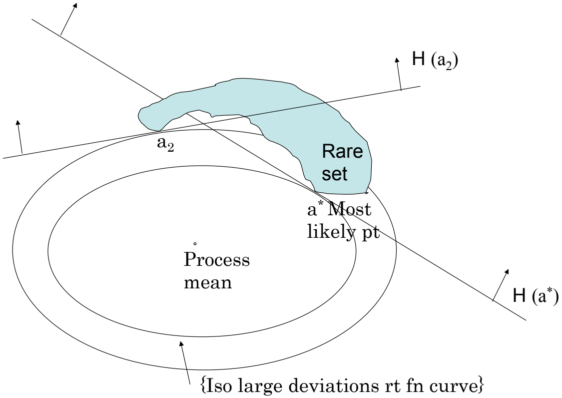
# Issues with importance Sampling




- First illustrated by Sadowsky and Bucklew (1991).

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- Developed closed form expressions for approximations to MLE.
- Conducted asymptotic analysis to prove effectiveness of proposed estimators and empirically verified strong performance relative to existing methods.
- If the underlying model is wrong (the only truth in this talk so far), the exact method and the approximate one are equally bad!
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