

# Option Pricing: A Review

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# Outline

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# Introduction

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- These notes review the principles underlying option pricing and some of the key concepts.
- One objective is to highlight the factors that affect option prices, and to see how and why they matter.
- We also discuss important concepts such as the option delta and its properties, implied volatility and the volatility skew.
- For the most part, we focus on the Black-Scholes model, but as motivation and illustration, we also briefly examine the binomial model.

# Outline of Presentation

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- The material that follows is divided into six (unequal) parts:
  - Options: Definitions, importance of volatility.
  - Pricing of options by replication: Main ideas, a binomial example.
  - The option delta: Definition, importance, behavior.
  - Pricing of options using risk-neutral probabilities.
  - The Black-Scholes model: Assumptions, the formulae, some intuition.
  - Implied Volatility and the volatility skew/smile.

# Definitions and Preliminaries

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- An *option* is a financial security that gives the holder the right to buy or sell a specified quantity of a specified asset at a specified price on or before a specified date.
  - Buy = *Call* option. Sell = *Put* option
  - On/before: *American*. Only on: *European*
  - Specified price = *Strike* or *exercise* price
  - Specified date = *Maturity* or *expiration* date
  - Buyer = holder = long position
  - Seller = writer = short position

# Options as Insurance

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- Options provide *financial insurance*.
  - The option holder has the right, but not the obligation, to participate in the specified trade.
- Example: Consider holding a put option on Cisco stock with a strike of \$25. (Cisco's current price: \$26.75.)
  - The put provides a holder of the stock with protection against the price falling below \$25.
- What about a call with a strike of (say) \$27.50?
  - The call provides a buyer with protection against the price increasing above \$25.

# The Option Premium

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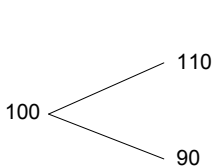
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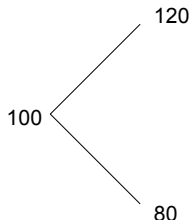
- The writer of the option provides this insurance to the holder.
- In exchange, writer receives an upfront fee called the *option price* or the *option premium*.
- Key question we examine: How is this price determined? What factors matter?

# The Importance of Volatility: A Simple Example

- Suppose current stock price is  $S = 100$ .
- Consider two possible distributions for  $S_T$ . In each case, suppose that the “up” and “down” moves each have probability  $1/2$ .



**Case 1: Low Vol**



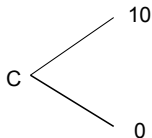
**Case 2: High Vol**

- Same mean but second distribution is more volatile.

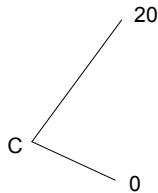


# Call Payoffs and Volatility

- Consider a call with a strike of  $K = 100$ .
- Payoffs from the call at maturity:



Call Payoffs: Low Vol

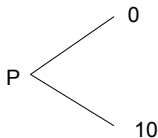


Call Payoffs: High Vol

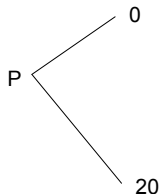
- The second distribution for  $S_T$  clearly yields superior payoffs.

# Put Payoffs and Volatility

- Puts similarly benefit from volatility. Consider a put with a strike of  $K = 100$ .
- Payoffs at maturity:



**Put Payoffs: Low Vol**



**Put Payoffs: High Vol**

- Once again, the second distribution for  $S_T$  clearly yields superior payoffs.

# Options and Volatility (Cont'd)

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- In both cases, the superior payoffs from high volatility are a consequence of “optionality.”
  - A forward with a delivery price of  $K = 100$  does not similarly benefit from volatility.
- Thus, all long option positions are also long volatility positions.
  - That is, long option positions increase in value when volatility goes up and decrease in value when volatility goes down.
- Of course, this means that all written option positions are short volatility positions.
- Thus, the amount of volatility anticipated over an option's life is a central determinant of option values.

# Put-Call Parity

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- One of the most important results in all of option pricing theory.
- It relates the prices of otherwise identical European puts and calls:

$$P + S = C + PV(K).$$

- Put-call parity is proved by comparing two portfolios and showing that they have the same payoffs at maturity.
  - **Portfolio A** Long stock, long put with strike  $K$  and maturity  $T$ .
  - **Portfolio B** Long call with strike  $K$  and maturity  $T$ , investment of  $PV(K)$  for maturity at  $T$ .

# Options and Replication

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- As with all derivatives, the basic idea behind pricing options is *replication*: we look to create identical payoffs to the option's using
  - Long/short positions in the underlying security.
  - Default-risk-free investment/borrowing.
- Once we have a portfolio that replicates the option, the cost of the option must be equal to the cost of replicating (or “synthesizing”) it.

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- As we have just seen, *volatility* is a primary determinant of option value, so we cannot price options without first modelling volatility.
- More generally, we need to model uncertainty in the evolution of the price of the underlying security.
- It is this dimension that makes option pricing more complex than forward pricing.
- It is also on this dimension that different “option pricing” models make different assumptions:
  - Discrete (“lattice”) models: e.g., the binomial.
  - Continuous models: e.g., Black-Scholes.

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- Once we have a model of prices evolution, options can be priced by replication:
  - Identify option payoffs at maturity.
  - Set up a portfolio to replicate these payoffs.
  - Value the portfolio and hence price the option.
- The replication process can be technically involved; we illustrate it using a simple example—a one-period binomial model.
- From the example, we draw inferences about the replication process in general, and, in particular, about the behavior of the option *delta*.
- Using the intuition gained here, we examine the Black-Scholes model.

# A Binomial Example

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- Consider a stock that is currently trading at  $S = 100$ .
- Suppose that one period from now, it will have one of two possible prices: either  $uS = 110$  or  $dS = 90$ .
- Suppose further that it is possible to borrow or lend over this period at an interest rate of 2%.
- What should be the price of a call option that expires in one period and has a strike price of  $K = 100$ ? What about a similar put option?



# Pricing the Call Option

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- We price the call in three steps:
  - First, we identify its possible payoffs at maturity.
  - Then, we set up a portfolio to replicate these payoffs.
  - Finally, we compute the cost of this replicating portfolio.
- Step 1 is simple: the call will be exercised if state  $u$  occurs, and not otherwise:
  - Call payoff if  $uS = 10$ .
  - Call payoff if  $dS = 0$ .

# The Call Pricing Problem

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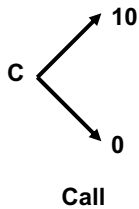
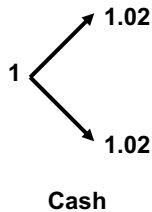
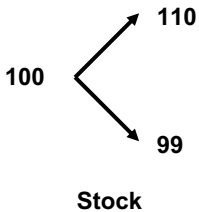
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# The Replicating Portfolio for the Call

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- To replicate the call, consider the following portfolio:
  - $\Delta_c$  units of stock.
  - $B$  units of lending/borrowing.
- Note that  $\Delta_c$  and  $B$  can be positive or negative.
  - $\Delta_c > 0$ : we are *buying* the stock.
  - $\Delta_c < 0$ : we are *selling* the stock.
  - $B > 0$ : we are *investing*.
  - $B < 0$ : we are *borrowing*.

# The Replicating Portfolio for the Call

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- For the portfolio to replicate the call, we must have:

$$\Delta_c \cdot (110) + B \cdot (1.02) = 10$$

$$\Delta_c \cdot (90) + B \cdot (1.02) = 0$$

- Solving, we obtain:

$$\Delta_c = 0.50 \quad B = -44.12.$$

- In words, the following portfolio perfectly replicates the call option:
  - A **long** position in 0.50 units of the stock.
  - **Borrowing** of 44.12.

# Pricing the Call (Cont'd)

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- Cost of this portfolio:  $(0.50) \cdot (100) - (44.12)(1) = 5.88$ .
- Since the portfolio perfectly replicates the call, we must have  $C = 5.88$ .
- Any other price leads to arbitrage:
  - If  $C > 5.88$ , we can sell the call and buy the replicating portfolio.
  - If  $C < 5.88$ , we can buy the call and sell the replicating portfolio.

# Pricing the Put Option

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- To replicate the put, consider the following portfolio:
  - $\Delta_p$  units of stock.
  - $B$  units of bond.
- It can be shown that the replicating portfolio now involves a **short** position in the stock and **investment**:
  - $\Delta_p = -0.50$ : a short position in 0.50 units of the stock.
  - $B = +53.92$ : investment of 53.92.
- As a consequence, the arbitrage-free price of the put is

$$(-0.50)(100) + 53.92 = 3.92.$$

# Summary

- Option prices depend on volatility.
- Thus, option pricing models begin with a description of volatility, or, more generally, of how the prices of the underlying evolves over time.
- Given a model of price evolution, options may be priced by replication.
  - Replicating a call involves a **long** position in the underlying and **borrowing**.
  - Replicating a put involves a **short** position in the underlying and **investment**.
- A key step in the replication process is identification of the option **delta**, i.e., the size of the underlying position in the replicating portfolio. We turn to a more detailed examination of the delta now.

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# The Option Delta

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- The *delta* of an option is the number of units of the underlying security that must be used to replicate the option.
- As such, the delta measures the “riskiness” of the option in terms of the underlying.
- For example: if the delta of an option is (say)  $+0.60$ , then, roughly speaking, the risk in the option position is the same as the risk in being long 0.60 units of the underlying security.
- Why “roughly speaking?”



# The Delta in Hedging Option Risk

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- The delta is central to **pricing** options by replication.
- As a consequence, it is also central to **hedging** written option positions.
  - For example, suppose we have written a call whose delta is currently  $+0.70$ .
  - Then, the risk in the call is the same as the risk in a long position in 0.70 units of the underlying.
  - Since we are **short** the call, we are essentially short 0.70 units of the underlying.
  - Thus, to hedge the position we simply buy 0.70 units of the underlying asset.
  - This is **delta hedging**.

# The Delta in Aggregating Option Risk

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- The delta enables us to aggregate option risk (on a given underlying) and express it in terms of the underlying.
- For example, suppose we have a portfolio of stocks on IBM stock with possibly different strikes and maturities:
  - Long 2000 calls (strike  $K_1$ , maturity  $T_1$ ), each with a delta of  $+0.48$ .
  - Long 1000 puts (strike  $K_2$ , maturity  $T_2$ ), each with a delta of  $-0.55$ .
  - Short 1700 calls (strike  $K_3$ , maturity  $T_3$ ), each with a delta of  $+0.63$ .
- What is the aggregate risk in this portfolio?

# The Delta in Aggregating Option Risk

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- Each of the first group of options (the strike  $K_1$ , maturity  $T_1$  calls) is like being **long** 0.48 units of the stock.
- Since the portfolio is long 2,000 of these calls, the aggregate position is akin to being long  $2000 \times 0.48 = 960$  units of the stock.
- Similarly, the second group of options (the strike  $K_2$ , maturity  $T_2$  puts) is akin to being **short**  $1000 \times 0.55 = 550$  units of the stock.
- The third group of options (the strike  $K_3$ , maturity  $T_3$  calls) is akin to being **short**  $1700 \times 0.63 = 1071$  units of the stock.
- Thus, the aggregate position is:  $+960 - 550 - 1071 = -661$  or a **short** position in 661 units of the stock.
- This can be delta hedged by taking an offsetting long position in the stock.

# The Delta as a Sensitivity measure

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- The delta is also a **sensitivity measure**: it provides a snapshot estimate of the dollar change in the value of a call for a given change in the price of the underlying.
- For example, suppose the delta of a call is  $+0.50$ .
- Then, holding the call is “like” holding  $+0.50$  units of the stock.
- Thus, a change of \$1 in the price of the stock will lead to a change of  $+0.50$  in the value of the call.

# The Sign of the Delta

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- In the binomial examples, the delta of the call was *positive*, while that of the put was *negative*.
- These properties must *always* hold. That is:
  - A long call option position is qualitatively like a long position in the underlying security.
  - A long put option position is qualitatively like a short position in the underlying security.
- Why is this the case?

# Maximum Value of the Delta

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- Moreover, the delta of a call must always be less than  $+1$ .
  - The maximum gain in the call's payoff per dollar increase in the price of the underlying is \$1.
  - Thus, we never need more than one unit of the underlying in the replicating portfolio.
- Similarly, the delta of a put must always be greater than  $-1$  since the maximum loss on the put for a \$1 increase in the stock price is \$1.

# Depth-in-the-Money (“Moneyness”) and the Delta

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- The delta of an option depends in a central way on the option's depth-in-the-money.
- Consider a call:
  - If  $S \gg K$  (i.e., is very high relative to  $K$ ), the delta of a call will be close to +1.
  - If  $S \ll K$  (i.e., is very small relative to  $K$ ), the delta of the call will be close to zero.
  - In general, as the stock price increases, the delta of the call will increase from 0 to +1.

# Moneyness and Put Deltas

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- Now, consider a put.
  - If  $S \gg K$ , the put is deep out-of-the-money. Its delta will be close to zero.
  - If  $S \ll K$ , the put is deep in-the-money. Its delta will be close to  $-1$ .
  - In general, as the stock price increases, the delta of the put increases from  $-1$  to  $0$ .



# Moneyness and Option Deltas

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- To summarize the dependence on moneyness:
  - The delta of a deep out-of-the-money option is close to zero.
  - The absolute value of the delta of a deep in-the-money option is close to 1.
  - As the option moves from out-of-the-money to in-the-money, the absolute value of the delta increases from 0 towards 1.
- The behavior of the call and put deltas are illustrated in the figure on the next page.

# The Option Deltas

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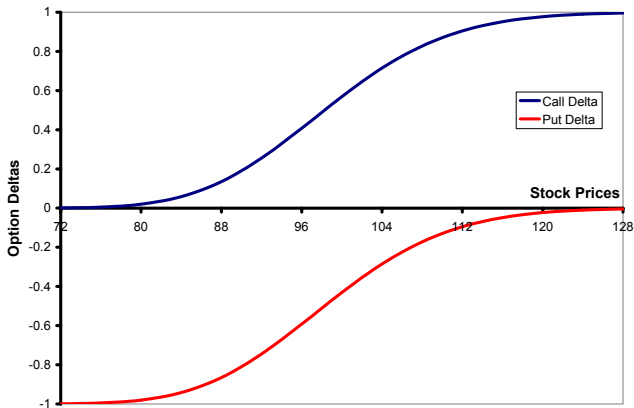
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# Implications for Replication/Hedging

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- The option delta's behavior has an important implication for option replication.
- In pricing forwards, “buy-and-hold” strategies in the spot asset suffice to replicate the outcome of the forward contract.
- In contrast, since an option's depth-in-the-money changes with time, so will its delta. Thus, a static buy-and-hold strategy will *not* suffice to replicate an option.
- Rather, one must use a *dynamic* replication strategy in which the holding of the underlying security is constantly adjusted to reflect the option's changing delta.

# Summary

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- The option delta measures the number of units of the underlying that must be held in a replicating portfolio.
- As such, the option delta plays many roles:
  - Replication.
  - (Delta-)Hedging.
  - As a sensitivity measure.
- The option delta depends on depth-in-the-money of the option:
  - It is close to unity for deep in-the-money options.
  - It is close to zero for deep out-of-the-money options.
  - Thus, it offers an intuitive feel for the probability the option will finish in-the-money.

# Risk-Neutral Pricing

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- An alternative approach to identifying the fair value of an option is to use the model's *risk-neutral* (or *risk-adjusted*) probabilities.
- This approach is guaranteed to give the same answer as replication, but is computationally a lot simpler.
- Pricing follows a three-step procedure.
  - Identify the model's risk-neutral probability.
  - Take expectations of the option's payoffs under the risk-neutral probability.
  - Discount these expectations back to the present at the risk-free rate.
- The number obtained in Step 3 is the fair value of the option.

# The Risk-Neutral Probability

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- The risk-neutral probability is that probability under which expected returns on all the model's assets are the same.
- For example, the binomial model has two assets: the stock which returns  $u$  or  $d$ , and the risk-free asset which returns  $R$ . ( $R = 1 +$  the risk-free interest rate.)
- If  $q$  and  $1 - q$  denote, respectively, the risk-neutral probabilities of state  $u$  and state  $d$ , then  $q$  must satisfy

$$q \cdot u + (1 - q) \cdot d = R.$$

- This means the risk-neutral probability in the binomial model is

$$q = \frac{R - d}{u - d}.$$

# Risk-Neutral Pricing: An Example

- Consider the one-period binomial example in which  $u = 1.10$ ,  $d = 0.90$ , and  $R = 1.02$ .

- In this case,

$$q = \frac{1.02 - 0.90}{1.10 - 0.90} = 0.60.$$

- Therefore, the expected payoffs of the call under the risk-neutral probability is

$$(0.60)(10) + (0.40)(0) = 6.$$

- Discounting these payoffs back to the present at the risk-free rate results in

$$\frac{6}{1.02} = 5.88,$$

which is the same price for the option obtained by replication.

# Risk-Neutral Pricing: A Second Example

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- Now consider pricing the put with strike  $K = 100$ .
- The expected payoff from the put at maturity is

$$(0.60)(0) + (0.40)(10) = 4.$$

- Discounting this back to the present at the risk-free rate, we obtain

$$\frac{4}{1.02} = 3.92,$$

which is the same price obtained via replication.

- As these examples show, risk-neutral pricing is computationally much simpler than replication.



# The Black-Scholes Model

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- The Black–Scholes model is unambiguously the best known model of option pricing.
- Also one of the most widely used: it is the benchmark model for
  - Equities.
  - Stock indices.
  - Currencies.
  - Futures.
- Moreover, it forms the basis of the Black model that is commonly used to price some interest-rate derivatives such as caps and floors.

# The Black-Scholes Model (Cont'd)

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- Technically, the Black-Scholes model is much more complex than discrete models like the binomial.
  - In particular, it assumes *continuous* evolution of uncertainty.
  - Pricing options in this framework requires the use of very sophisticated mathematics.
- What is gained by all this sophistication?
  - Option prices in the Black-Scholes model can be expressed in *closed-form*, i.e., as particular explicit functions of the parameters.
  - This makes computing option prices and option sensitivities very easy.

# Assumptions of the Model

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- The main assumption of the Black-Scholes model pertains to the evolution of the stock price.
- This price is taken to evolve according to a **geometric Brownian motion**.
- Shorn of technical details, this says essentially that two conditions must be met:
  - Returns on the stock have a **lognormal** distribution with constant volatility.
  - Stock prices cannot jump (the market cannot “gap”).

# The Log-Normal Assumption

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- The log-normal assumption says that the (natural) **log** of returns is **normally** distributed: if  $S$  denotes the current price, and  $S_t$  the price  $t$  years from the present, then

$$\ln\left(\frac{S_t}{S}\right) \sim N(\mu t, \sigma^2 t).$$

- Mathematically, **log-returns** and **continuously-compounded returns** represent the same thing:

$$\ln\left(\frac{S_t}{S}\right) = x \Leftrightarrow \frac{S_t}{S} = e^x \Leftrightarrow S_t = Se^x.$$

- Thus, log-normality says that returns on the stock, expressed in continuously-compounded terms, are normally distributed.

# Volatility

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- The number  $\sigma$  is called the *volatility* of the stock. Thus, volatility in the Black-Scholes model refers to the standard deviation of annual log-returns.
- The Black-Scholes model takes this volatility to be a constant. In principle, this volatility can be estimated in two ways:
  - From historical data. (This is called *historical volatility*.)
  - From options prices. (This is called *implied volatility*.)
- We discuss the issue of volatility estimation in the last part of this segment. For now, assume the level of volatility is known.

# Is GBM a Reasonable Assumption?

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- The log-normality and no-jumps conditions appear unreasonably restrictive:
  - Volatility of markets is typically not constant over time.
  - Market prices do sometimes “jump.”
  - In particular, the no-jumps assumption appears to rule out dividends.
- Dividends (and similar predictable jumps) are actually easily handled by the model.
- The other issues are more problematic, and not easily resolved. We discuss them in the last part of this presentation.

# Option Prices in the Black-Scholes Model

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- Option prices in the Black–Scholes model may be recovered using a replicating portfolio argument.
- Of course, the construction—and maintenance—of a replicating portfolio is significantly more technically complex here than in the binomial model.
- We focus here instead on the final option prices that result and the intuitive content of these prices.

# Notation

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- $t$ : current time.
- $T$ : Horizon of the model. (So time-left-to-maturity:  $T - t$ .)
- $K$ : strike price of option.
- $S_t$ : current price of stock.
- $S_T$ : stock price at  $T$ .
- $\mu, \sigma$ : Expected return and volatility of stock (annualized).
- $r$ : risk-free rate of interest.
- $C, P$ : Prices of call and put (European only).



# Call Option Prices in the Black-Scholes Model

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- The call-pricing formula in the Black-Scholes model is

$$C = S_t \cdot N(d_1) - e^{-r(T-t)} K \cdot N(d_2)$$

where  $N(\cdot)$  is the *cumulative* standard normal distribution [ $N(x)$  is the probability under a standard normal distribution of an observation less than or equal to  $x$ ], and

$$d_1 = \frac{1}{\sigma\sqrt{T-t}} \left[ \ln\left(\frac{S_t}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t) \right]$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

- This formula has a surprisingly simple interpretation.

# Call Prices and Replication

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- To replicate a call in general, we must
  - Take a long position in  $\Delta_c$  units of the underlying, and
  - Borrow  $B_c$  at the risk-free rate.
- The cost of this replicating portfolio—which is the call price—is

$$C = S_t \cdot \Delta_c - B_c. \quad (1)$$

- The Black-Scholes formula has an identical structure: it too is of the form

$$C = S_t \times [\text{Term 1}] - [\text{Term 2}]. \quad (2)$$

# Call Prices and Replication (Cont'd)

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- Comparing these structures suggests that

$$\Delta_c = N(d_1). \quad (3)$$

$$B_c = e^{-r(T-t)}K \cdot N(d_2). \quad (4)$$

- This is exactly correct! The Black-Scholes formula is obtained precisely by showing that the composition of the replicating portfolio is (3)–(4) and substituting this into (1).
- In particular,  $N(d_1)$  is just the delta of the call option.

# Put Option Prices in the Black–Scholes Model

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- Recall that to replicate a put in general, we must
  - Take a *short* position in  $|\Delta_p|$  units of the underlying, and
  - Lend  $B_p$  at the risk-free rate.
- Thus, in general, we can write the price of the put as

$$P = B_p + S_t \Delta_p \quad (5)$$

- The Black-Scholes formula identifies the exact composition of  $\Delta_p$  and  $B_p$ :

$$\Delta_p = -N(-d_1) \quad B_p = PV(K) N(-d_2)$$

where  $N(\cdot)$ ,  $d_1$ , and  $d_2$  are all as defined above.

# Put Prices in the Black–Scholes Model (Cont'd)

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- Therefore, the price of the put is given by

$$P = PV(K) \cdot N(-d_2) - S_t \cdot N(-d_1). \quad (6)$$

- Expression (6) is the Black-Scholes formula for a European put option.
- Equivalently, since  $N(x) + N(-x) = 1$  for any  $x$ , we can also write

$$P = S_t \cdot [N(d_1) - 1] + PV(K) \cdot [1 - N(d_2)] \quad (7)$$

# Black-Scholes via Risk-Neutral Probabilities

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- Alternative way to derive Black–Scholes formulae: use risk-neutral (or risk-adjusted) probabilities.
- To identify option prices in this approach: take expectation of terminal payoffs under the risk-neutral probability measure and discount at the risk-free rate.
- The terminal payoffs of a call option with strike  $K$  are given by

$$\max\{S_T - K, 0\}.$$

- Therefore, the arbitrage-free price of the call option is given by

$$C = e^{-rT} E_t^* [\max\{S_T - K, 0\}]$$

where  $E_t^*[\cdot]$  denotes time- $t$  expectations under the risk-neutral measure.

# Black-Scholes via Risk-Neutral Probs (Cont'd)

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- Equivalently, this expression may be written as

$$C = e^{-rT} E_t^* [S_T - K \mid S_T \geq K]$$

- Splitting up the expectation, we have

$$C = e^{-rT} E_t^* [S_T \mid S_T \geq K] - e^{-rT} E_t^* [K \mid S_T \geq K].$$

- From this to the Black-Scholes formula is simply a matter of grinding through the expectations, which are tedious, but not otherwise difficult.

# Black-Scholes via Risk-Neutral Probs (Cont'd)

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- Specifically, it can be shown that

$$e^{-rT} E_t^* [S_T \mid S_T \geq K] = S_t N(d_1)$$

$$e^{-rT} E_t^* [K \mid S_T \geq K] = e^{-rT} K N(d_2)$$

- In particular,  $N(d_2)$  is the risk-neutral probability that the option finishes in-the-money (i.e., that  $S_T \geq K$ ).



# Remarks on the Black-Scholes Formulae

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- Two remarkable features of the Black–Scholes formulae:
  - Option prices only depend on *five* variables:  $S$ ,  $K$ ,  $r$ ,  $T$ , and  $\sigma$ .
  - Of these five variables, only *one*—the volatility  $\sigma$ —is not directly observable.
- This makes the model easy to implement in practice.

# Remarks (Cont'd)

- It must also be stressed that these are *arbitrage-free* prices.
  - That is, they are based on construction of replicating portfolios that perfectly mimic option payoffs at maturity.
  - Thus, if prices differ from these predicted levels, the replicating portfolios can be used to create riskless profits.
- The formulae can also be used to delta-hedge option positions.
  - For example, suppose we have written a call option whose current delta, using the Black-Scholes formula, is  $N(d_1)$ .
  - To hedge this position, we take a long position in  $N(d_1)$  units of the underlying.
  - Of course, dynamic hedging is required.

# Remarks (Cont'd)

- Finally, it must be stressed again that closed-form expressions of this sort for option prices is a rare occurrence.
  - The particular advantage of having closed forms is that they make computation of option sensitivities (or option “greeks”) a simple task.
- Nonetheless, such closed-form expressions exist in the Black-Scholes framework only for *European*-style options.
  - For example, closed-forms do not exist for American put options.
  - However, it *is* possible to obtain closed-form solutions for certain classes of exotic options (such as compound options or barrier options).

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# Plotting Black-Scholes Option Prices

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- Closed-forms make it easy to compute option prices in the Black-Scholes model:

- 1 Input values for  $S_t$ ,  $K$ ,  $r$ ,  $T-t$ , and  $\sigma$ .
- 2 Compute  $d_1 = [\ln(S_t/K) + (r + \sigma^2/2)(T-t)]/[\sigma\sqrt{T-t}]$ .
- 3 Compute  $d_2 = d_1 - \sigma\sqrt{T-t}$ .
- 4 Compute  $N(d_1)$ .
- 5 Compute  $N(d_2)$ .
- 6 Compute option prices.

$$C = S_t N(d_1) - e^{-r(T-t)} K N(d_2)$$

$$P = e^{-r(T-t)} K [1 - N(d_2)] - S_t [1 - N(d_1)]$$

# Plotting Option Prices (Cont'd)

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- The following figure illustrates this procedure.

- Four parameters are held fixed in the figure:

$$K = 100, T - t = 0.50, \sigma = 0.20, r = 0.05$$

- The figures plot call and put prices as  $S$  varies from 72 to 128.
- Observe non-linear reaction of option prices to changes in stock price.
  - For deep OTM options, slope  $\approx 0$ .
  - For deep ITM options, slope  $\approx 1$ .
- Of course, this slope is precisely the option delta!

# Plotting Option Prices

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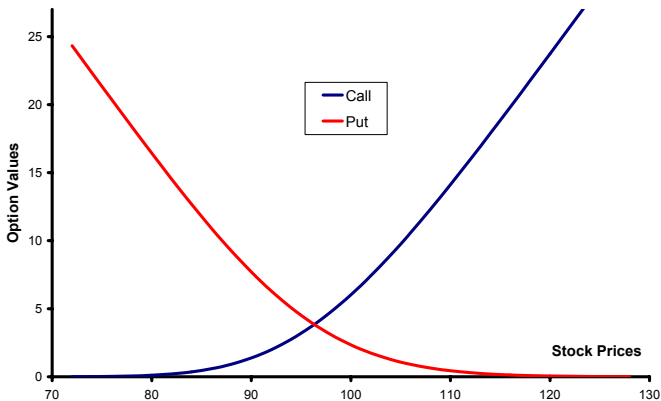
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# Implied Volatility

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- Given an option price, one can ask the question: what level of volatility is implied by the observed price?
- This level is the *implied volatility*.
- Formally, implied volatility is the volatility level that would make observed option prices consistent with the Black-Scholes formula, given values for the other parameters.

# Implied Volatility (Cont'd)

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- For example, suppose we are looking at a call on a non-dividend-paying stock.
- Let  $K$  and  $T-t$  denote the call's strike and time-to-maturity, and let  $\hat{C}$  be the call's price.
- Let  $S_t$  be the stock price and  $r$  the interest rate.
- Then, the implied volatility is the unique level  $\sigma$  for which

$$C^{bs}(S, K, T-t, r, \sigma) = \hat{C},$$

where  $C^{bs}$  is the Black-Scholes call option pricing formula.

- Note that implied volatility is uniquely defined since  $C^{bs}$  is strictly increasing in  $\sigma$ .



# Implied Volatility (Cont'd)

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- Implied volatility represents the “market’s” perception of volatility anticipated over the option’s lifetime.
- Implied volatility is thus forward looking.
- In contrast, historical volatility is backward looking.

# The Volatility Smile/Skew

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- In theory, any option (any  $K$  or  $T$ ) may be used for measuring implied volatility.
- Thus, if we fix maturity and plot implied volatilities against strike prices, the plot should be a flat line.
- In practice, in equity markets, implied volatilities for “low” strikes (corresponding to out-of-the-money puts) are typically much higher than implied volatilities for ATM options. This is the *volatility skew*.
- In currency markets (and for many individual equities), the picture is more symmetric with way-from-the-money options having higher implied volatilities than at-the-money options. This is the *volatility smile*.
- See the screenshots on the next 4 slides.

# S&P 500 Implied Volatility Plot

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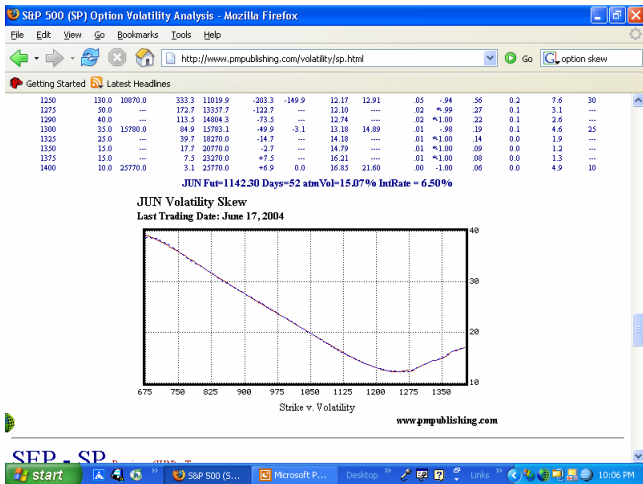
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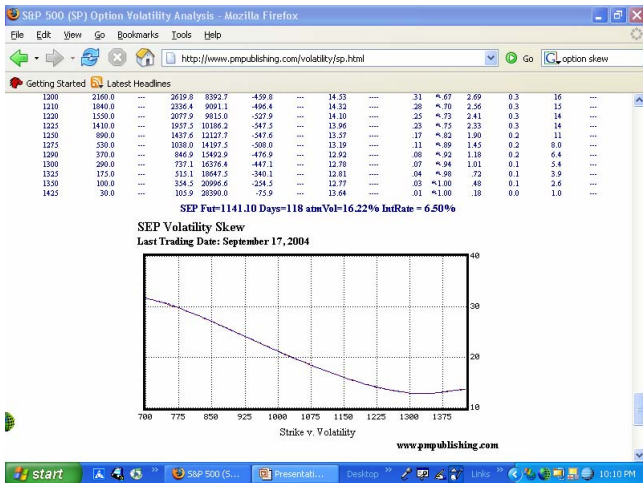
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# USD-GBP Implied Volatility Plot

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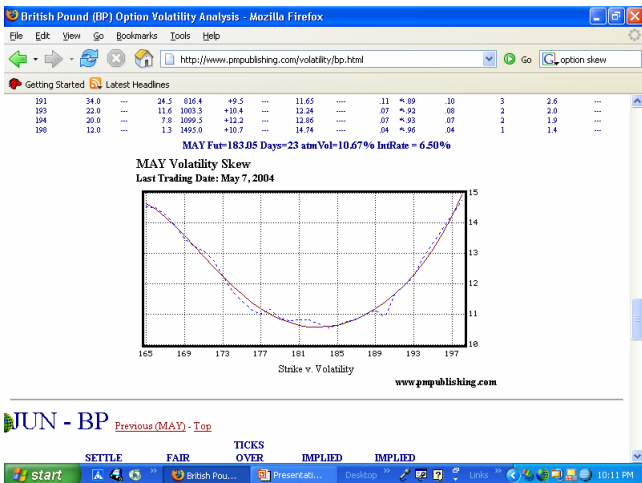
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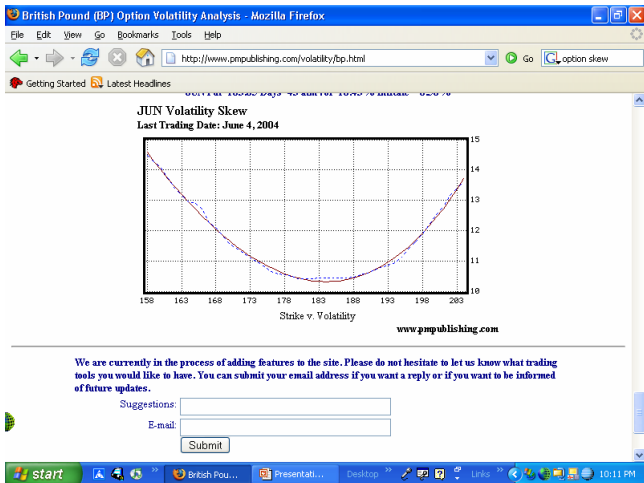
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# The Source of the Volatility Skew

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- Two sources are normally ascribed for the volatility skew.
- One is the returns distribution. The Black-Scholes model assumes log-returns are normally distributed.
- However, in every financial market, extreme observations are far more likely than predicted by the log-normal distribution.
  - Extreme observations = observations in the *tail* of the distribution.
  - Empirical distributions exhibit “fat tails” or *leptokurtosis*.
- Empirical log-returns distributions are often also *skewed*.

# Source of the Volatility Skew (Cont'd)

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- Fat tails  $\implies$  Black-Scholes model with a constant volatility will *underprice* out-of-the-money puts relative to those at-the-money.
- Put differently, “correctly” priced out-of-the-money puts will reflect a higher implied volatility than at-the-money options.
- This is exactly the volatility skew! That is, the volatility skew is evidence not only that the lognormal model is not a fully accurate description of reality but also that the market recognizes this shortcoming.
- As such, there is valuable information in the smile/skew concerning the “actual” (more accurately, the market's expectation) of the return distribution. For instance:
  - More symmetric smile  $\implies$  Less skewed distribution.
  - Flatter smile/skew  $\implies$  Smaller kurtosis.



# Source of the Volatility Skew (Cont'd)

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- The other reason commonly given for the volatility skew is that the world as a whole is net long equities, and so there is a positive net demand for protection in the form of puts.
- This demand for protection, coupled with market frictions, raises the price of out-of-the-money puts relative to those at-the-money, and results in the volatility skew.
- With currencies, on the other hand, there is greater symmetry since the world is net long both currencies, so there is two-sided demand for protection.
- Implicit in this argument is Rubinstein's notion of "crash-o-phobia," fear of a sudden large downward jump in prices.

# Generalizing Black-Scholes

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- Obvious question: why not generalize the log-normal distribution?
- Indeed, there may even be a “natural” generalization.
- The Black-Scholes model makes two uncomfortable assumptions:
  - No jumps.
  - Constant volatility.
- If jumps are added to the log-normal model or if volatility is allowed to be stochastic, the model will exhibit fat tails and even skewness.

# Generalizing Black-Scholes (Cont'd)

- So why don't we just do this?
  - We can, but complexity increases (how many new parameters do we need to estimate?).
  - Jumps & SV have very different dynamic implications.
  - Reality is more complex than either model (indeed, substantially so).
- Ultimately: better to use a simple model with known shortcomings?
- On jumps vs stochastic volatility models, see
  - Das, S.R. and R.K. Sundaram (1999) "Of Smiles and Smirks: A Term-Structure Perspective," Journal of Financial and Quantitative Analysis 34(2), pp.211-239.*

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