Use of Control Variate Technique: Structured Credit Index Products

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Credit Risk Elective

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Estimated Tranche Spreads – Base Case

Number of credits (N)	125
CDS Spread (uniform, basis points)	100
Maturity (years)	3
Riskless rate (rf)	0.05
LGD (percentage)	0.6
Change in CDS for delta basis points (del_CDS)	20
Number of trials in MC	500,000

		Spread	
Attachment	***Detachment***	***Corr = 0.25***	***Corr = 0.5***
0 %	3%	1705	1176
3%	7%	609	542
7%	10%	247	324
10%	15%	102	201
15%	30%	16	76
30%	100%	0.1	3.0

Calculating the Delta

$$\begin{array}{c}\text{Change in MTM}\\ \text{of Tranche}\end{array} = (S' - S) \quad \$250 \quad (\beta_2 \; - \beta_1) \quad \frac{1}{r}(1 \quad e^{-3r})$$

$$\begin{array}{c}\text{Change in MTM}\\ \text{of CDS (20 bp shift)}\end{array} = (120 \text{ bp - } 100 \text{ bp}) \quad \frac{\$250}{125} \quad \frac{1}{r}(1 \; - e^{-3r})$$

$$\begin{array}{c}\text{Delta of Tranche}\\ \text{per 1 bp shift}\end{array} = \frac{\text{Change in Mark-to-Market of Tranche}}{\text{Change in Mark-to-Market of Credit}}$$

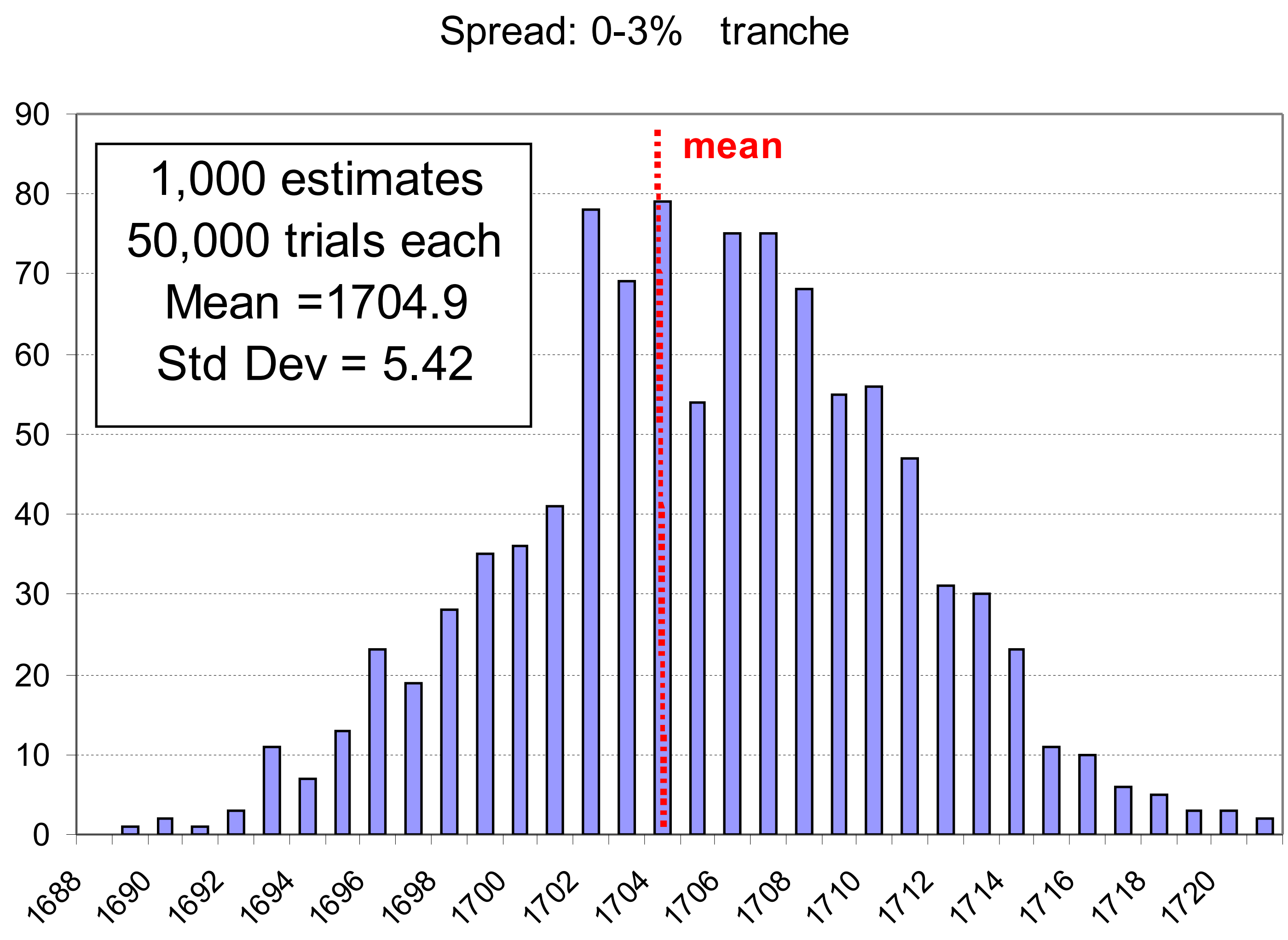
$$= 125 \quad (\beta_2 \; - \beta_1) \quad \frac{(S' - S)}{20bp}$$

Note: S is the tranche spread in the base case and S' is the spread after a 20 bp shift in the CDS spread on one credit

Delta

		Delta	
Attachment	***Detachment***	***Corr = 0.25***	***Corr = 0.5***
0 %	3%	28.9	23.4
3%	7%	30.5	21.3
7%	10%	12.3	10.9
10%	15%	10.0	13.4
15%	30%	6.1	16.5
30%	100%	0.2	4.1

How Accurate is the Estimated Spread?



Impact of Error in Spread on Accuracy of the delta

Example – Equity Tranche

- Suppose ***spreads*** are measured with an ***error*** of ε and that we measure the spreads ***independently*** (no control variate):

For tranche spread: $\delta = 125 \times (\beta_2 - \beta_1) \times \frac{(S'-S)}{20bp} = 0.1875 \times (S' - S)$

$$\hat{S} = S + \varepsilon \qquad \hat{S}' = S' + \varepsilon' \quad (S:\ \text{true value};\ \hat{S}:\ \text{estimate})$$

True delta: $\delta = 0.1875 \times (S' - S)$

estimated delta: $\hat{\delta} = 0.1875 \times (\hat{S}' - \hat{S})$

Error in delta: $\hat{\delta} - \delta = 0.1875 \times (\varepsilon' - \varepsilon)$

$$\text{Variance}(\hat{\delta}) = 0.1875^2 \times \left[Var(\varepsilon') + Var(\varepsilon) - 2Cov(\varepsilon', \varepsilon) \right]$$

$$= 0.1875^2 \times 2 \times Var(\varepsilon) \times (1 - \rho_\varepsilon), \quad \rho_\varepsilon = \text{corr}(\varepsilon', \varepsilon)$$

Impact of Error in Spread on Accuracy of the delta, cntd. Example – *Equity Tranche*

- If the MC trials for the calculation of *S’* and *S* are independent then the correlation ρ_{ε} between ε’ and ε is zero and:

$$\text{Variance}(\hat{\delta}) = 0.1875^2 \quad 2 \quad Var(\varepsilon) = 2.06, \quad \sigma_{\varepsilon} = 5.42$$

$$\text{Std Dev}(\delta) = 1.42 \;\; \text{(i.e., 142\%)}$$ <<<when actual value of delta = 29%!!!

Estimates of Delta from Independent Estimates of the Spread are Highly Inaccurate

	Tranche					
	0-3	***3-7***	***7-10***	***10-15***	***15-30***	***30-100***
Delta (%)	2 8	30.5 9	12.3	10.0	6.1	0.2
	No control Variate					
Width	3%	4%	3%	5%	15%	70%
Multiplier	0.1875	0.2500	0.1875	0.3125	0.9375	4.3750
SD Spread (N = 50,000)	5.42	4.74	3.41	2.23	0.72	0.02
Predicted SD Delta (%)	***143.6***	***167.6***	***90.5***	***98.6***	***95.6***	***14.5***
Ratio SD/Level	***5.0***	***5.5***	***7.4***	***9.9***	***15.7***	***72.4***

- With independent estimates of the spread, the estimates of delta have standard errors that are between 5 and 70 times as large as delta itself

Brute Force: More Trials in the MC Simulation?

- The true delta for the equity tranche is about 29%
  - ✓ suppose we would like to measure this with a standard error of, say, 2%
- With independent MC estimates and 50,000 trials the standard error is around 142% with, around 70 times too large
- To improve the standard error by a factor of 70 would mean increasing the number of trials by a factor of (approximately) $70^2 = 4900$, i.e., from 50,000 trials to 245,000,000!!!

A better way

- To reduce the error, we need to make the error in the spread in (a) the base case and (b) the "shifted" case ***positively correlated***.

$$\text{Variance}(\hat{\delta}) = 0.1875^2 \quad 2 \quad Var(\varepsilon) \quad (1 \quad \rho), \qquad \rho = \text{corr}\,(\varepsilon', \varepsilon)$$

- To achieve this we use the ***same set of random variables*** to calculate the spread in both the base case and the "shifted" case

A Better Way

- Table shows error in the equity tranche spread estimate without and with control variate
- ***With control variate***, error has a standard deviation that is 100 times smaller!

	Without Control Variate			With Control Variate		
	base	***w. shift***	***difference***	***base***	***w. shift***	***difference***
1	-1.24	-4.13	2.90	-11.01	-9.51	-1.50
2	2.96	5.56	-2.60	-5.31	-3.71	-1.60
3	-3.74	9.96	-13.70	4.89	6.49	-1.60
4	10.17	-3.13	13.30	-7.91	-6.41	-1.50
5	-2.34	-0.74	-1.60	-1.11	0.49	-1.60
6	-5.24	4.96	-10.20	3.89	5.29	-1.40
7	-7.54	17.26	-24.80	6.19	7.59	-1.40
8	3.66	0.37	3.30	-2.31	-0.71	-1.60
9	0.06	-10.94	11.00	-0.11	1.39	-1.50
10	6.96	-10.14	17.10	8.99	10.39	-1.40
11	6.26	-1.63	7.90	-6.01	-4.41	-1.60
12	-8.54	8.16	-16.70	-3.31	-1.81	-1.50
13	1.96	0.26	1.70	-2.91	-1.41	-1.50
14	0.66	1.16	-0.50	1.39	2.89	-1.50
15	1.76	-3.13	4.90	6.29	7.89	-1.60
16	6.06	-3.74	9.80	5.99	7.49	-1.50
17	-2.63	0.16	-2.80	-3.41	-1.71	-1.70
18	-4.54	3.56	-8.10	-5.61	-4.31	-1.30
19	-6.54	-7.13	0.60	-0.01	1.69	-1.70
20	-1.44	-3.54	2.10	-3.91	-2.41	-1.50
Std. Dev	5.17	6.78	10.43	5.41	5.40	0.10

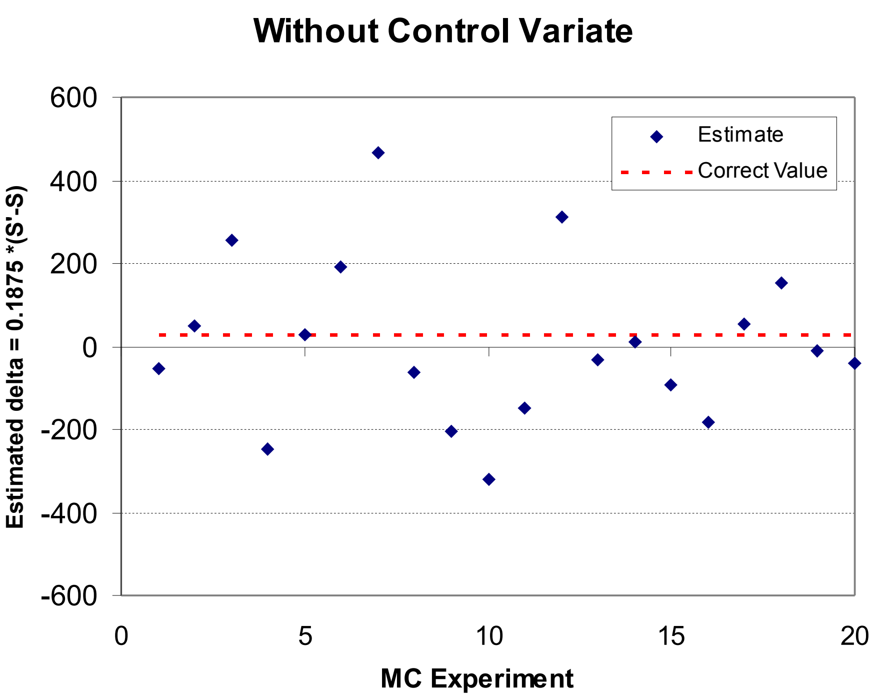
Spread Estimates with and Without Control Variate

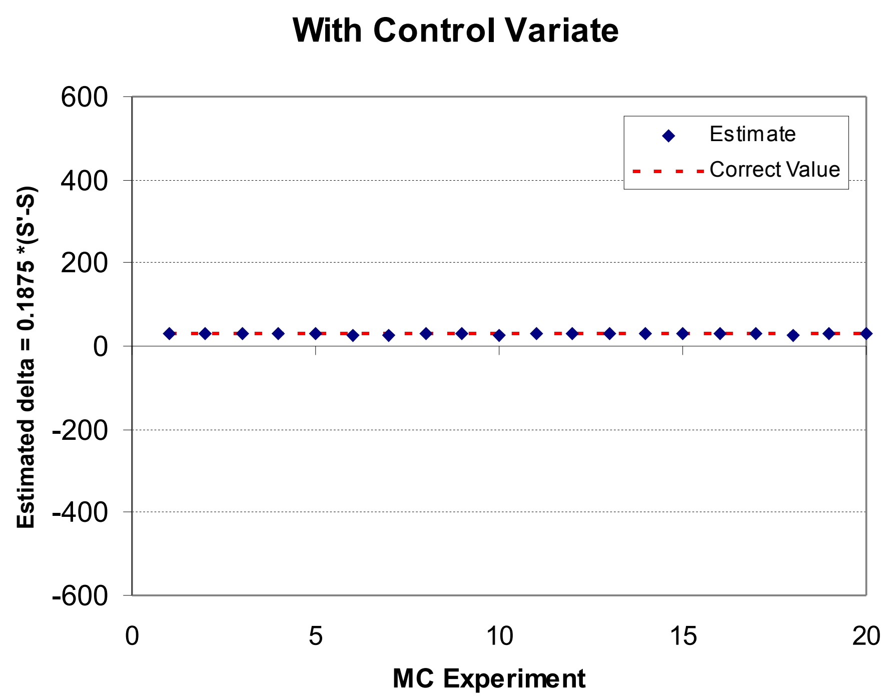
- Estimating delta as equivalent to estimating the difference between *S'* and *S*

$$\hat{\delta} = 0.1875 \quad (\hat{S}' - \hat{S}) \qquad \hat{S} = S + \varepsilon \qquad \hat{S}' = S' + \varepsilon'$$

Estimates of delta without and with control variate technique

(20 estimates, 50,000 trials each)





Use of Control Variate Technique Improves Precision of Delta Substantially

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Ratio SD/Level	***5.0***	***5.5***	***7.4***	***9.9***	***15.7***	***72.4***
	With control Variate					
SD Spread difference	0.10	0.08	0.06	0.04	0.01	0.00
Predicted SD Delta (%)	***1.91***	***2.05***	***1.10***	***1.29***	***0.87***	***0.27***
Predicted Ratio SD/Level	***0.07***	***0.07***	***0.09***	***0.13***	***0.14***	***1.37***
Actual Ratio SD/Level	***0.05***	***0.07***	***0.09***	***0.13***	***0.15***	***1.02***