Merton-model Approach to Valuing Correlation Products

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Binomial with Merton Model

- Important method for calculating distribution of loan losses:
  - ✓ widely used in banking
  - ✓ used by Basel II regulations to set bank capital requirements

- Linked to distance-to-default analysis

Mixed Binomial: Using Merton's Models as Mixing Distribution

- In Merton model value of risky debt depends on ***firm value*** and ***default risk is correlated because firm values are correlated*** (e.g., via common dependence on ***market factor***).
- Value of firm i at time T:

$$V_{T,i} = V_i \exp\Big(\underbrace{(\mu_i - (1/2)\sigma^2_{V,i})T}_{\text{expected value of } R^C} + \underbrace{\sigma_{V,i}\sqrt{T}\varepsilon_i}_{\text{surprise in } R^C} \Big) \quad \text{where } \varepsilon_i \sim N(0,1)$$

- We will assume that ***correlation between firm values*** arises because of correlation between surprise in individual firm value (ε_i) and ***market factor (m)***

Mixed Binomial: Using Merton's Models as Mixing Distribution

- Suppose correlation between each firm's value and the market factor is the same and equal to sqrt(ρ) .
- This means that we may model correlation between the ε's as

$$\varepsilon_i = \sqrt{\rho} m + \sqrt{1-\rho} v_i, \quad i = 1,\ldots, N$$

and

$$corr(\varepsilon_i, \varepsilon_j) = \rho$$

- Where *m and* v_i are independent *N(0,1)* random variables and ρ is common to all firms
- Notice that if $v_i \sim N(0,1)$ and $m \sim N(0,1)$ then $\varepsilon_i \sim N(0,1)$

Structural Approach, contd.

- From our analysis of ***distance-to-default***, we know that under the Merton Model a firm defaults when:

$$\varepsilon_i \; - \; R_{D,i} \quad (\mu_i \quad \frac{1}{2}\sigma_{V,i}^2)T \quad / \sigma_{V,i}\sqrt{T} \text{ where } R_{D,i} \quad \ln(B_i / V_i)$$

- The ***unconditional*** probability of default, *p*, is therefore:

$$p \;\; <\text{Prob} \;\; \varepsilon_i \quad \frac{R_{D,i} - (\mu_i - \frac{1}{2}\sigma_{V,i}^2)}{\sigma_{V,i}\sqrt{T}} = N \;\; \frac{R_D - (\mu_i - \frac{1}{2}\sigma_{V,i}^2)}{\sigma_i\sqrt{T}}$$

- In this model we assume that the ***default probability***, *p*, is ***constant across firms***

Structural Approach to Correlation – the Idea

- Working out the distribution of portfolio losses directly when the ε's are correlated is not easy
- But, if we work out the ***distribution conditional on the market shock***, m, then we can exploit the fact that the remaining shocks are independent and work out the portfolio loss distribution

Structural Approach, contd.

- The shock to the return, ε_i, is related to the common and idiosyncratic shocks by:

$$\varepsilon_i = \sqrt{\rho} m + \sqrt{1-\rho} v_i$$

- Default occurs when:

$$\varepsilon_i = \sqrt{\rho} m + \sqrt{1-\rho} v_i < \frac{R_{D,i} - (\mu_i - \frac{1}{2}\sigma_{V,i}^2)}{\sigma_{V,i}\sqrt{T}} = N^{-1}(p)$$

or

$$v_i < \frac{N^{-1}(p) - \sqrt{\rho} m}{\sqrt{1-\rho}}$$

The Default Condition

$$v_i < \frac{N^{-1}(p) - \sqrt{\rho}m}{\sqrt{1-\rho}}$$

- A large value of *m* means a "good" shock to the market (high asset values)
- The larger the value of *m* the more negative the idiosyncratic shock, v_i, has to be to trigger default
- The higher the correlation, ρ , between the firm shocks, the larger the impact of *m* on the critical value of v_i.

Structural Approach, contd.

- Conditional on the realisation of the common shock, *m*, the probability of default is therefore:

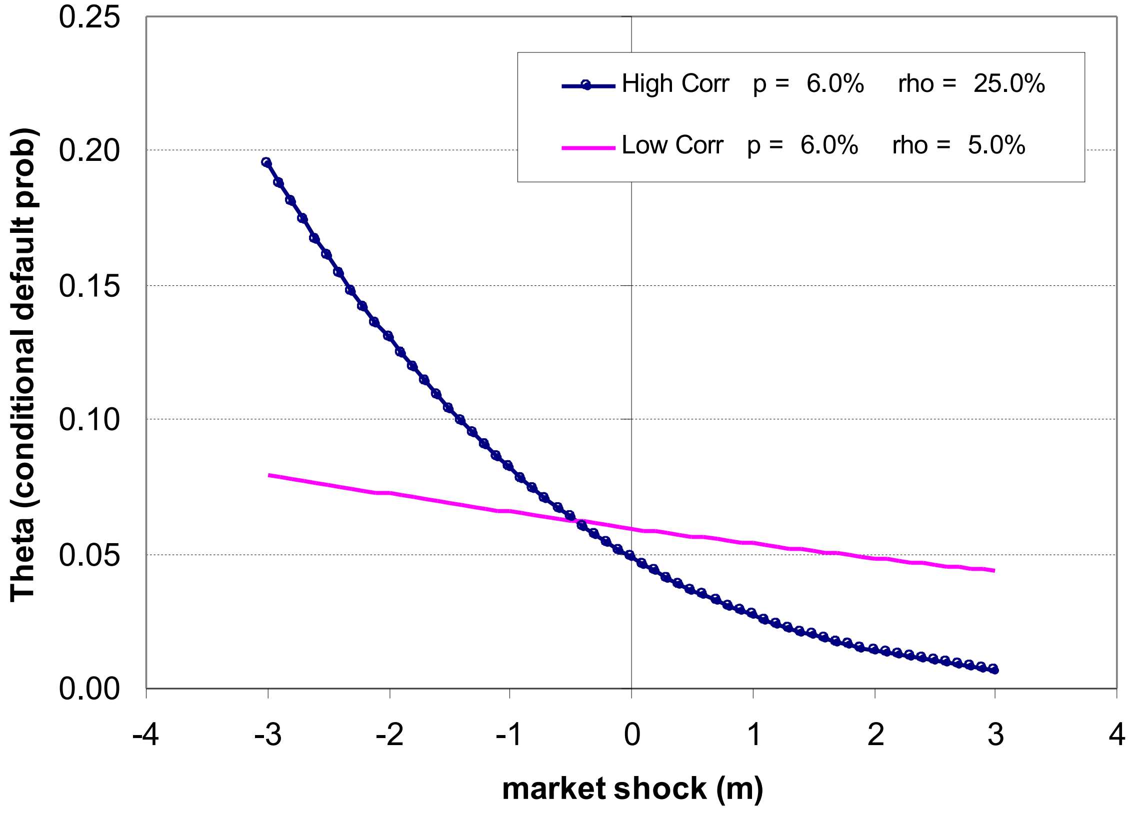
$$\text{Prob}(\text{default}\,|\,m) = \text{Prob}\left(v_i < \frac{N^{-1}(p) - \sqrt{\rho}m}{\sqrt{1-\rho}} \right)$$

$$= N\left(\frac{N^{-1}(p) - \sqrt{\rho}m}{\sqrt{1-\rho}} \right) = \theta(m), \text{ say}$$

$$\text{and therfore } = \frac{N^{-1}(p) - \sqrt{\rho}m}{\sqrt{1-\rho}} = N^{-1}(\theta)$$

The relation between *m* and θ

- For a given market shock, *m*, *θ* gives the conditional probability of default on an individual loan



Implications of Conditional Independence

- For a given value of m, as the number of loans in the portfolio $\rightarrow \infty$, the ***proportion*** of loans in the portfolio that default converges ***to the probability θ***
- The probability that the loan-loss proportion, L, is $< \theta$ is therefore:

$$\text{Prob}(L = \theta) \quad \text{Prob} \quad \frac{N^{-1}(p)-\sqrt{\rho}m}{\sqrt{1-\rho}} \quad N^{-1}(\theta)$$

$$= \text{Prob} \quad m \quad \frac{1}{\sqrt{\rho}}\left(N^{-1}(p) - N^{-1}(\theta)\sqrt{1-\rho} \right)$$

$$= \text{Prob} \quad m \quad -\frac{1}{\sqrt{\rho}}\left(\sqrt{1-\rho}\, N^{-1}(\theta) \quad N^{-1}(p) \right)$$

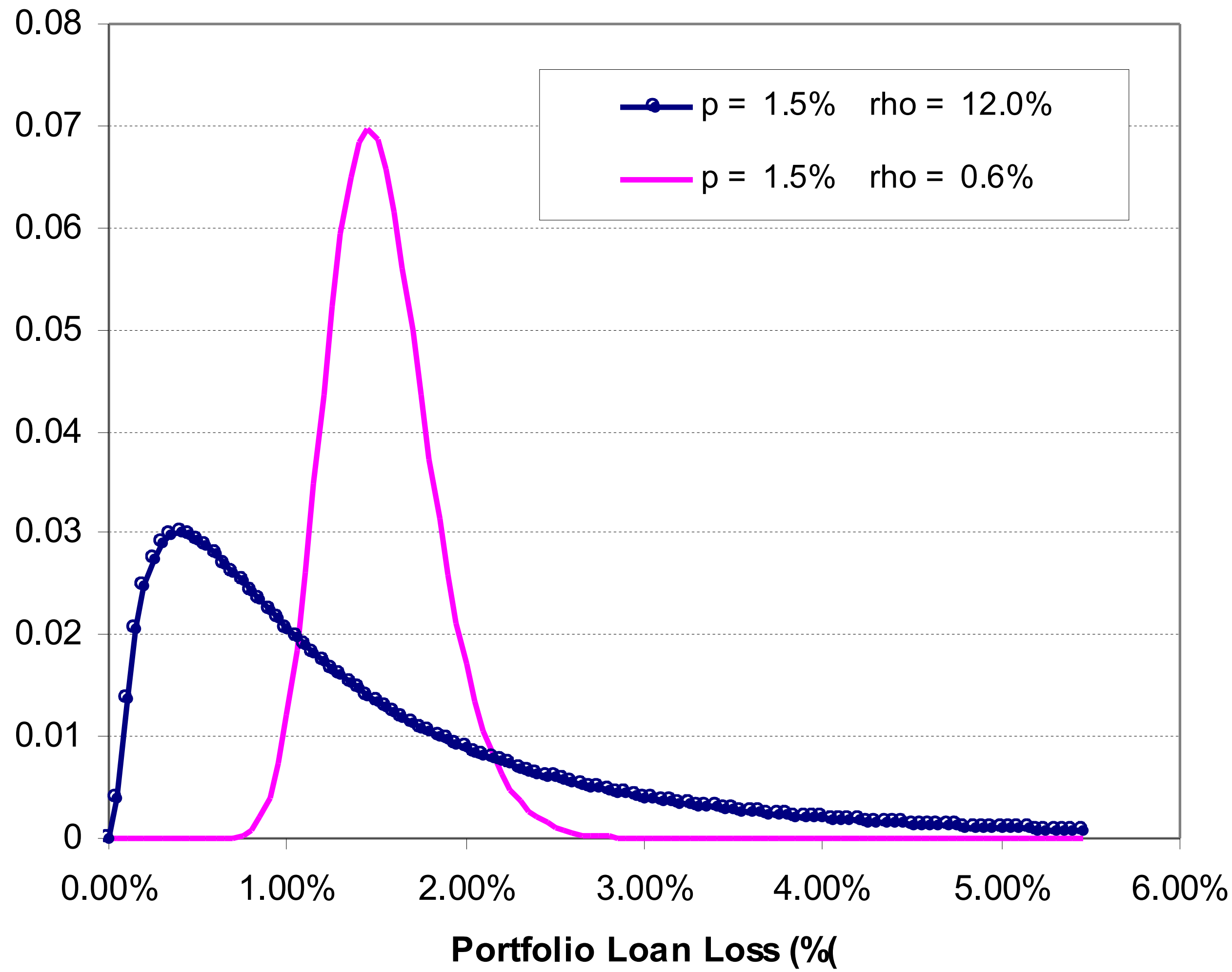
$$= \text{N} \quad \frac{1}{\sqrt{\rho}}\left(\sqrt{1-\rho}\, N^{-1}(\theta) - N^{-1}(p) \right)$$

Loan Loss Distribution – Structural Model

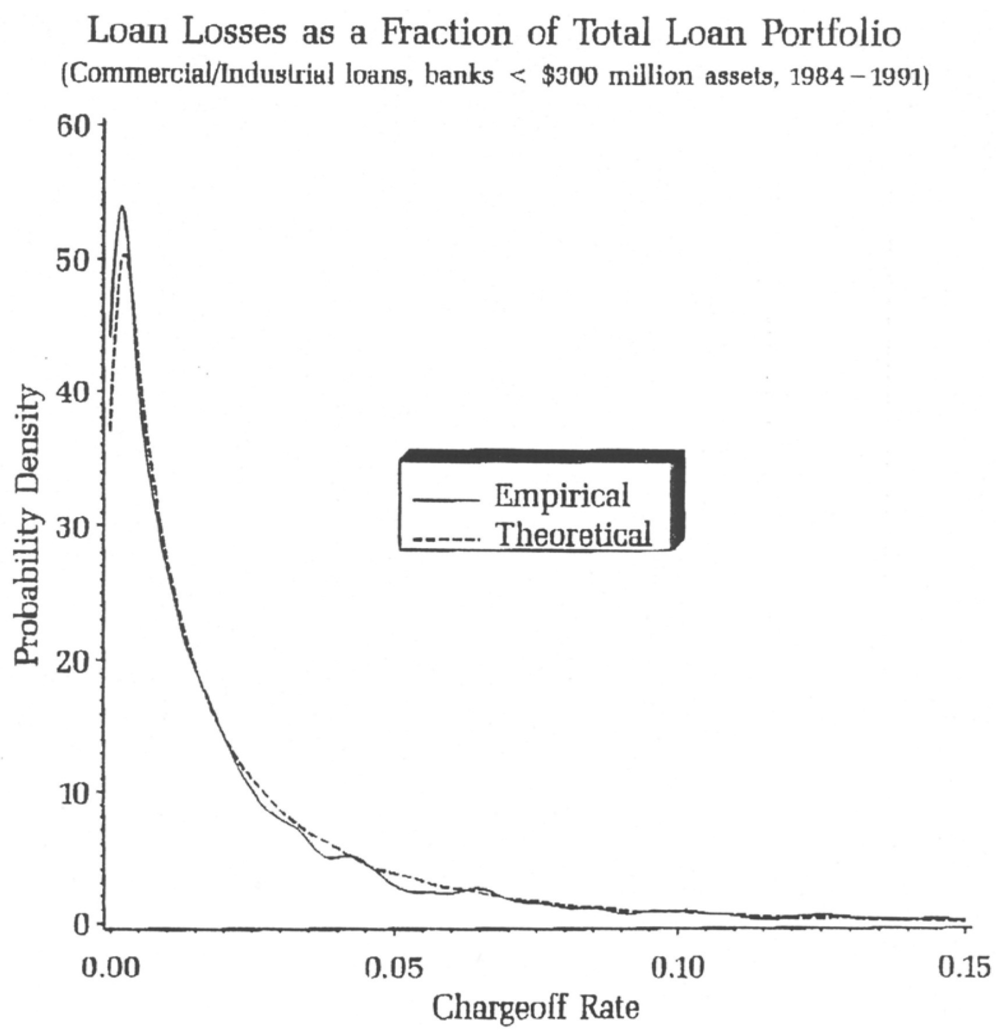
$$\text{Prob}(L = \theta) \quad N\left(\frac{1}{\sqrt{\rho}}\left(\sqrt{1 \quad \rho}\, N^{-1}(\theta) \quad N^{-1}(p)\right)\right)$$

- This result gives the distribution of the ***fraction of loans that default*** in a ***well diversified homogeneous portfolio*** where the correlation in default comes from dependence on a ***common factor***
- ***Homogeneity*** means that each loan has:
  - ✓ the ***same default probability***, p
  - ✓ (implicitly) the ***same loss-given-default***
  - ✓ the ***same correlation***, ρ , across different loans
- The distribution has ***two parameters***
  - ✓ default probability, p
  - ✓ correlation, ρ

Loan Loss Distribution with p = 1% and ρ = 12% and 0.6%



Example of Vasicek formula Applied to Bank Portfolio



Source: Vasicek