The Copula Approach to Valuing Correlation Products

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Modelling Correlation:

- ***Default*** is a ***binomial*** event: it happens or it doesn't
- With a fixed recovery rate the distribution of portfolio losses is the distribution of the ***number*** of ***defaults***
- But ***difficult*** to include default ***correlation*** directly into standard binomial framework
- Two common approaches:
  - ✓ Copula Approach
    - ➢ widely used in pricing – but needs caution
  - ✓ Structural Approach: sounder approach... in future?

Caution over Copulas

- The copula approach means is that we can ***always separate*** the ***dependence structure*** between two or more random variables from their ***unconditional*** (or marginal) ***distributions***.
- Sounds very powerful ***BUT*** problem is that often very ***little guidance*** available in how to choose copula
- ***Gaussian copula*** (method described here) ***widely used*** in practice but quite possibly a ***poor description*** of reality.

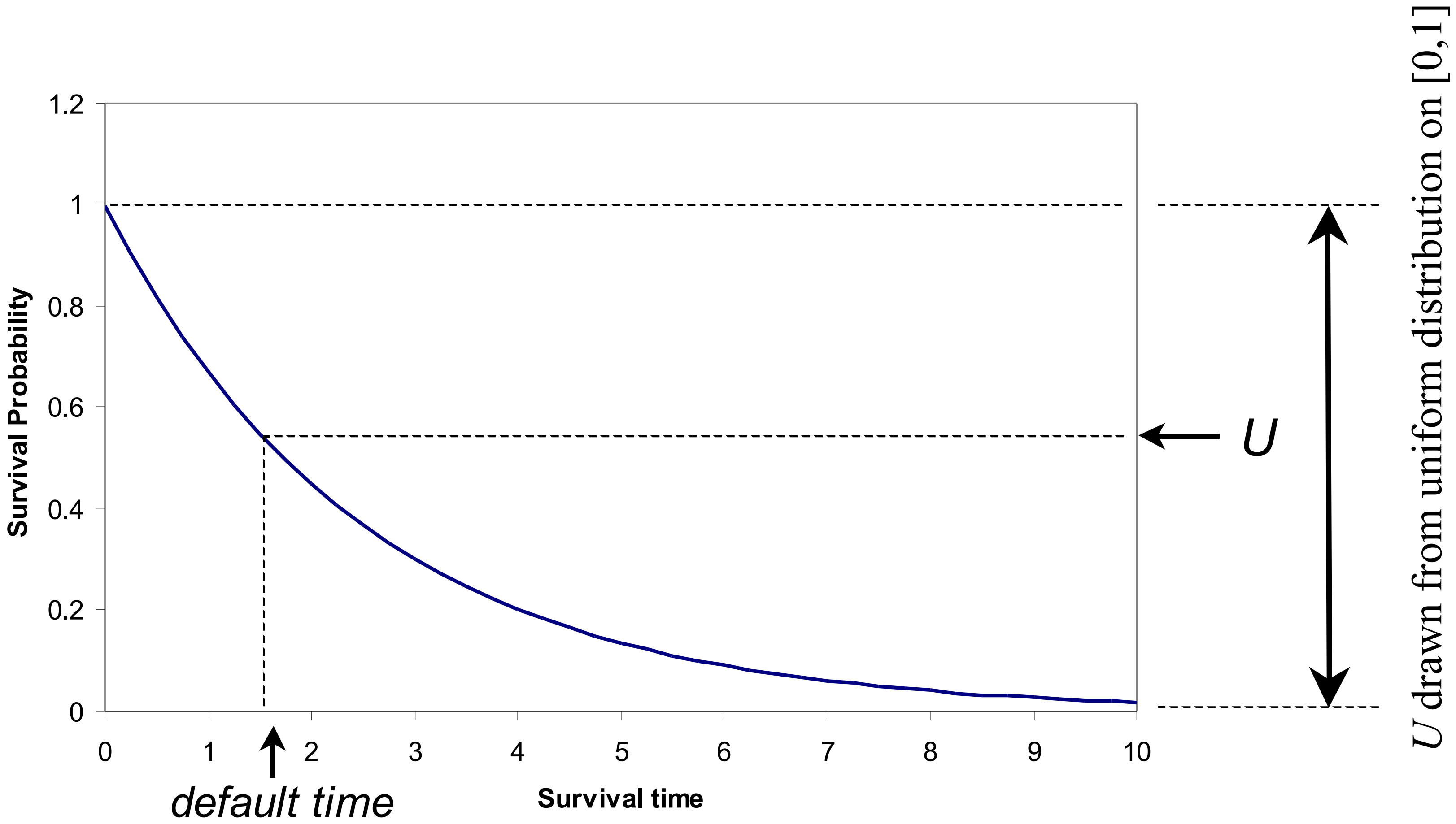
Simulating Default Times

- The starting point is the ***intensity model*** with constant intensity λ
- Under this model the ***probability of survival*** up to time t is:

$$p(\tau > t) = \exp(-\lambda t)$$

- As with any cumulative distribution, if we were to make a random drawing from the distribution of default times, τ , the cumulative probability $p(\tau > t)$ would be equally likely to be anywhere within the range zero to one (see further intuition below).

Simulating default time in Intensity Model



Inverse Cumulative Method for Random Numbers: Intuition

See diagram on next but one slide

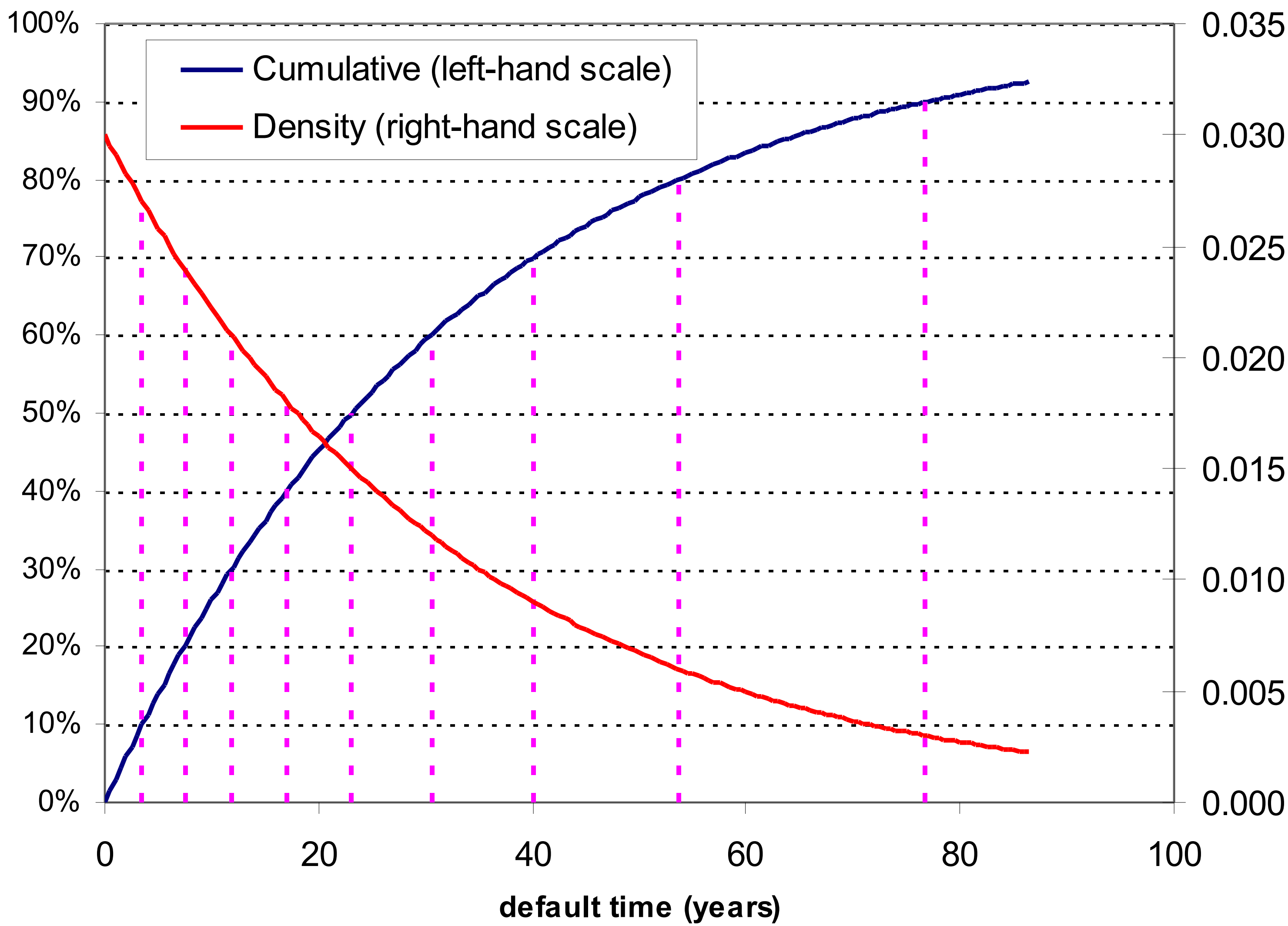
- We wish to make a random drawing from the distribution of default times
- The diagram shows both the probability density and the cumulative distribution.
- The total area under the density is ***one***: suppose we divide up this area into 10 equal regions (marked by the vertical dotted lines)
- A default time drawn at random would be equally likely to fall into any of these 10 intervals
- We now use the following rule:
  - ✓ randomly draw a number between 1 and 10
  - ✓ use this to choose one of the 10 intervals
  - ✓ our random number is the value of the default time in the middle (say) of the interval.

continued next slide

Inverse Cumulative Method: Contd.

- All that is required to implement this method is to know where the boundaries of the intervals lie.
- With 10 intervals each interval accounts for 10% of the probability and so the cumulative probability at the first boundary is 10%, at the second it is 20% and so forth.
- ***We can simply look up these values on the cumulative distribution:*** notice that the default time boundaries for the probability density (horizontal axis) correspond to the 10%, 20% etc. points on the cumulative distribution.
- We could therefore implement the method as follows:
  - ✓ Choose a number (***k***) from one to 1 to 10
  - ✓ Look up the value of the default times that corresponds to cumulative probabilities of (k -1)* 10% and k*10% (the left and right hand boundaries for the k^{th} interval) and choose the number in the middle.
- The actual method we use (choosing U from a uniform distribution on [0,1]) is equivalent to doing this with an infinite number of intervals

Inverse Cumulative Method for Random Numbers: Intuition



Simulating Default Times

- In summary, therefore, to simulate a default time τ in the intensity model we:
  1. choose a random number, U, so that it is equally likely to be anywhere in the range $\{0,1\}$ – i.e., from a uniform distribution on $[0,1]$.
  2. solve :

$$U = \exp(-\lambda\tau) \qquad \tau = -\frac{1}{\lambda}\ln(U)$$

     and the value τ of we obtain is a random drawing from the distribution of default times

The Gaussian Copula Method for Default-Time Correlation and FTD Valuation

- To simulate ***correlated default times*** for FTD and CDO valuation an approach known as the ***Gaussian Copula Method*** is often used
- Correlation is modelled either through dependence on a ***single common factor*** or (sometimes) from a general correlation matrix
- Using the single common factor approach: if the correlation between each ***pair of names*** is ρ then for N names we calculate correlated random variables $\varepsilon_1, \ldots \varepsilon_N$ as:

$$\varepsilon_i = \sqrt{\rho} m + \left(\sqrt{1-\rho} \right) v_i, \quad i = 1, \hat{\mathrm{W}}, N, \quad m \sim N(0,1) \text{ and } v_i \sim N(0,1)$$

Note: m and v_i and v_i and v_j are idependent and therefore

$$corr(\varepsilon_i, \varepsilon_j) = \operatorname{cov}(\varepsilon_i, \varepsilon_j) = \rho$$

Generating Correlated Default Times

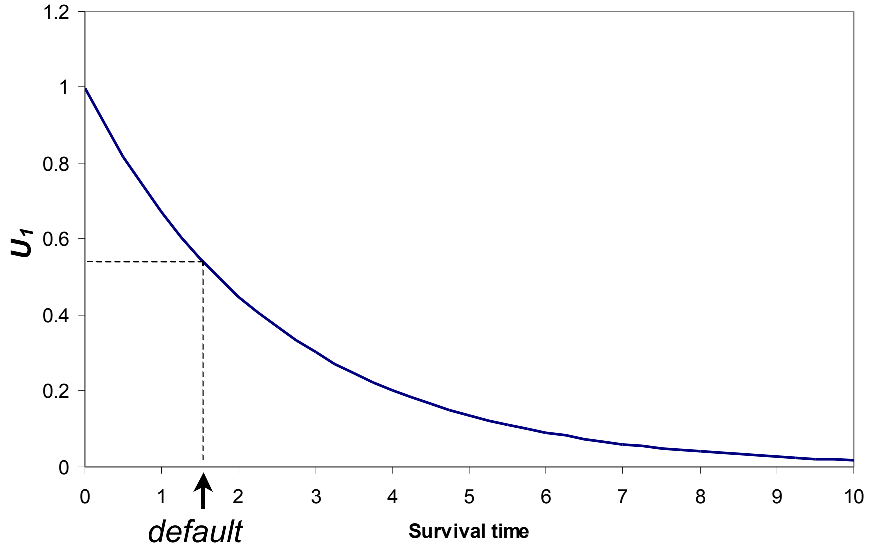
- For each trial in the simulation:
  - ✓ generate N correlated values of ε (as on previous slide) – one for each name/credit
  - ✓ for each of the ε's, calculate the corresponding default time as:

$$\tau_i = -\frac{1}{\lambda}\ln(U_i) \quad \text{where } U_i = N(\varepsilon_i)$$

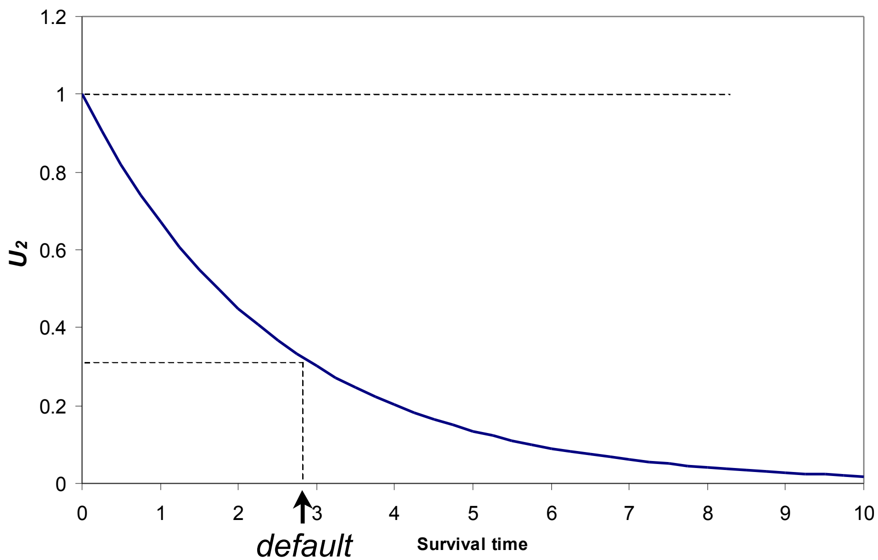
  and *N(.)* represents the cumulative normal distribution

- For an FTD, calculate the ***minimum*** time-to-default and, if this is less than the contract maturity record a ***default***

Generating Correlated Default Times



- Simulated correlated U_1, U_2 etc. .. And use these to generate correlated default times



Valuing an FTD – The Basic Idea

- Using simulation
  1. value loss leg up to time of default or end of contract, which ever comes first
  2. value premium leg for 1 b.p. – again, up to time of default or end of contract, which ever comes first
  3. find premium that equates value of loss and premium legs

Valuing an FTD

- Value of the ***loss leg*** of the FTD
  - ✓ expected discounted value of the loss leg
- Value of the ***premium leg*** (for a 1 b.p. fee, for example)
  - ✓ Expected discounted value of the 1 b.p. fee stream to default or maturity, whichever is shorter
- Dividing the value of the loss leg by the value of a 1 b.p. per year premium leg ,
- we obtain the FTD premium.

- As already noted: sqrt(ρ) is ***correlation*** between each ***firm*** and ***market*** and so ***correlation*** between each ***pair of firms is ρ*** .

Next week … CDOs, tranches etc.