Litterman and Iben Model of Estimating Credit Risk[1]

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Outline

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6. Modeling Default Likelihood as Intensities: An Introduction

1. Litterman and Iben (1991) model: The setup

- First paper to model credit risk in a reduced-form model.
- The model uses three inputs:
  1. Current term-structure of riskless bond yields.
  2. Current term-structure of risky bond yields.
  3. A model for the evolution of riskless interest rates.
- Using these inputs, the model derives:
  1. Forward default probabilities for the risky bond: for each t, given no default up to $t-1$, what is the probability of default in period t?
  2. Evolution of the term-structure of risky bond yields.
- Paper discusses the second issue only informally.

1.1 Recovery Rates in the Litterman-Iben Model

- Recovery rates are *not* an input in the Litterman-Iben model because the paper assumes 100% losses in the event of default.

- However, we will present a generalized version of their model in which positive recovery rates are admitted.

- Specifically, we will make the Recovery of Treasury (RT) assumption:
  - The recovery rate is denoted by δ ($0 \leq \delta \leq 1$).
  - In words, this means whenever the risky bond defaults, the holder of the risky bond receives delta units of a riskless bond with the same maturity (i.e., is guaranteed a payoff of $\delta\times$ face-value of risky bond at maturity).
  - For simplicity, we assume constant recovery rates through the model.
  - This could easily be changed to allow recovery rates to depend on time.

- We will also illustrate how to change the model for Recovery of Par or Face Value assumption.

1.2 Notation and Preliminaries

- The face value of all bonds is normalized to 1.

- We will denote by $r(t)$ the riskless t–year yield, and by $B(t)$ the price of a riskless t–year zero. Of course:

$$B(t) = \frac{1}{(1 + r(t))^t}.$$

- Similarly, $r^*(t)$ and $B^*(t)$ will denote, respectively, the risky t–year yield and the price of a risky t–year zero:

$$B^*(t) = \frac{1}{(1 + r^*(t))^t}.$$

- $s(t)$ will denote the t–year spread; by definition $r^*(t) = r(t) + s(t)$.

Notation & Preliminaries (Cont'd)

- For the model of evolution of the term-structure, we will use the Black-Derman-Toy model of short rate evolution.

- The analysis proceeds in two steps:

  1. First, the term-structures of risky and riskless bonds are used to identify the forward probabilities of default. *The model of riskless interest-rate evolution plays no role here.*
  2. Then, these probabilities of default are combined with the evolution of riskless rates to derive the evolution of the risky term-structure. This could be required, for example, to price credit derivatives such as spread options.

- All references to probabilities in the sequel are to *risk-neutral probabilities*.

- Finally, we illustrate the ideas in a numerical example. The input information is on the next two pages.

Notation & Preliminaries (Cont'd)

We will use the following input information on the initial term-structures:

Year	Riskless Yields	Spread	Risky Yields	Riskless Bond Prices	Risky Bond Prices
1	10.00	0.50	10.50	90.91	90.50
2	11.00	0.55	11.55	81.16	80.36
3	12.00	0.60	12.60	71.18	70.05
4	12.50	0.65	13.15	62.43	61.01
5	13.00	0.70	13.70	54.28	52.63

Recall that the BDT model calibrates interest-rates to both the current risk-free yields and to yield volatility. For a refresher on BDT, see accompanying slides on BDT model by Raghu Sundaram, NYU.

Notation & Preliminaries (Cont'd)

Secondly, we use the short rate tree from the Black-Derman-Toy (BDT) paper:

Year 1	Year 2	Year 3	Year 4	Year 5
				25.53
			21.79	
		19.42		19.48
	14.32		16.06	
10		13.77		14.86
	9.79		11.83	
		9.76		11.34
			8.72	
				8.65

Note: In all periods, "up" and "down" moves are equally likely.

2. Identifying the Forward Probabilities of Default

- Consider a one-year risky bond.
- At the end of one-year, the bond pays:

$$\begin{cases} 1, & \text{if no default} \\ \delta, & \text{if default} \end{cases}$$

- Let p_1 be the (risk-neutral) probability of default in one year.
- The probability of survival during year 1 is equal to $q_1 = 1 - p_1$.
- Then, expected return on the bond:

$$\frac{q_1 \cdot 1 + p_1 \cdot \delta}{B^*(1)}.$$

Forward Default Probabilities (Cont'd)

- Under the risk-neutral probability, the expected return on all assets should be the same.

- Therefore, we must have:

$$\frac{q_1 \cdot 1 + p_1 \cdot \delta}{B^*(1)} = \frac{1}{B(1)}.$$

- Solving, we obtain

$$p_1 = \left(\frac{1}{1-\delta}\right)\left(1 - \frac{B^*(1)}{B(1)}\right).$$

- Assume $\delta = 0.4$.

- In our example, this results in: $p_1 = 0.00754$ and $q_1 = 0.99246$.

Forward Default Probabilities (Cont'd)

- Now, consider the conditional probability of default in period 2, given no default in period 1.
- Call this probability p_2.
- The probability of survival during first two years is now

$$q_2 = (1 - p_1) \cdot (1 - p_2) = q_1 \cdot (1 - p_2).$$

- Consider a two-year risky bond. The payoffs from the bond at maturity are:

Event	Payoff	Probability
Default in Period 1	δ	p_1
Default in Period 2	δ	$q_1 \cdot p_2$
No Default	1	q_2

Forward Default Probabilities (Cont'd)

- Therefore, the expected return on the risky two-year bond is

$$\left(\frac{q_2 \cdot 1 + q_1 p_2 \cdot \delta + p_1 \cdot \delta}{B^*(2)}\right)^{1/2}$$

- This must equal the return on the riskless bond $(1/B(2))^{1/2}$, which gives

$$q_2 \cdot 1 + q_1 p_2 \cdot \delta + p_1 \cdot \delta = \frac{B^*(2)}{B(2)}$$

- Since p_1 is known, we can easily solve for p_2.

- In our example, this gives us $p_2 = 0.00892$ and $q_2 = 0.98361$.

Forward Default Probabilities (Cont'd)

- Using this procedure, we can solve for all the forward probabilities of default p_i and corresponding survival probabilities $q_i = (1 - p_1) \ldots (1 - p_i) = q_{i-1}(1 - p_i)$:

$$q_i \cdot 1 + q_{i-1} p_i \cdot \delta + q_{i-2} p_{i-1} \cdot \delta + \cdots + q_1 p_2 \cdot \delta + p_1 \cdot \delta = \frac{B^*(i)}{B(i)}$$

- In our example, the final result is:

Year	Probability of Default	Probability of Survival
1	0.00754	0.99246
2	0.00892	0.98361
3	0.01028	0.97350
4	0.01178	0.96203
5	0.01321	0.94932

Putting the Litterman-Iben model in perspective

- Let us take a step back and reflect on what we have achieved so far.
- We have identified risk-neutral default probabilities that are consistent with the current term-structures of risk-free and risky interest rates.
- Any credit derivative that can be replicated with risk-free and risky bonds can now be priced using these RNP's of default.
- The exercise is one of numerically fitting a given reduced-form model to the set of market data.
- In particular, we are not concerned with whether risky spreads are compensation just for default risk or also for other factors like liquidity.
- Even if spreads are not 100% about default risk, our RNP's of default remain the same *as long as these other factors do not affect the replication of credit derivatives we wish to price.*

3. Pricing Credit Default Swaps

- Consider a 5-year credit default swap.

- Assume the CDS fee S_{RT} is paid annually (starting at year 1) and the payments terminate at maturity or upon default whichever occurs first.
  - Note that this is unlike a standard contract where the fee would be paid quarterly. This is easy to incorporate.
  - We will focus on $1 notional. The standard contract is usually for USD 10 mln.

- Assume also that default may occur only at the end of a year in which case the protection buyer still pays that year's fee.

- The protection buyer receives the underlying bond in case of default. Under Recovery of Treasury assumption, this is equivalent to receiving a payment of $(1-\delta)$ units of a riskless bond that matures in the end of year 5.

- What should the par CDS fee S_{RT} be to get the credit default swap priced at zero?

Pricing CDS (Cont'd)

- Value of the Protection Leg of the CDS is:

$$(1-\delta)\cdot(p_1+q_1p_2+q_2p_3+q_3p_4+q_4p_5)\cdot B(5)=0.0165$$

- Here, for example, the third term $(1-\delta)q_2p_3B(5)$ is the expected present value of what the protection buyer receives if default occurs in the end of year 3.
  - q_2p_3 is the likelihood of default in year 3, which is survival probability till year 2 times the conditional likelihood of default in year 3.
  - $(1-\delta)$ is the recovery of treasury, so it is the payment in year 5.
  - $B(5)$ is the 5-year discount factor.

- Assume for now the CDS fee is 1bp.

- Value of the Premium Leg of the CDS is:

$$(B(1)+q_1B(2)+q_2B(3)+q_3B(4)+q_4B(5))\cdot 1bp=3.545\cdot 10^{-4}$$

Pricing CDS (Cont'd)

- Above we assumed CDS fee equal to 1 bp.

- Therefore, the value of the Premium Leg for our CDS is $3.545 \cdot S_{RT}$, where the fee S_{RT} is expressed in basis points per notional.

- Since the par CDS fee S_{RT} should make the value of Protection Leg equal to the value of the Premium Leg, we get (setting $q_0 = 1$):

$$S_{RT} = 10^4 \cdot (1 - \delta) \cdot \frac{\sum_{i=1}^{5} q_{i-1} p_i B(5)}{\sum_{i=1}^{5} q_{i-1} B(i)}$$

- For our specific example,

$$S_{RT} = 10^4 \cdot \frac{0.0165}{3.545} = 46.56\ bp.$$

- What is S for one year swap? Ignoring the 10^4 factor, it has the intuitive form:

$$S = (1 - \delta) p_1.$$

Pricing CDS (Cont'd)

- Consider now the same 5-year CDS contract except that, if default occurs, the protection buyer receives a fixed payment of \$1 at the time of default.
  - Note that this is like a digital default option.
  - Also note that it does NOT protect the buyer against recovery risk.
- Then, other things unchanged, the value of the Protection Leg of CDS is:

$$1 \cdot (p_1 B(1) + q_1 p_2 B(2) + q_2 p_3 B(3) + q_3 p_4 B(4) + q_4 p_5 B(5)) = 0.03529$$

- Thus, the par CDS fee S_2 expressed in basis points per notional is given by:

$$S_D = 10^4 \cdot \frac{\sum_{i=1}^{5} q_{i-1} p_i B(i)}{\sum_{i=1}^{5} q_{i-1} B(i)} = 10^4 \cdot \frac{0.03529}{3.545} = 99.56 \; bp.$$

Pricing Credit Default Swaps (Recovery of Par)

- Consider now the more natural recovery assumption, the Recovery of Par (RP).
- This differs from RT assumption. Under RP, if default occurs, the protection buyer receives the bond whose recovery value is $(1-\delta)$ *at the time of default*.
- Then, the value of the Protection Leg of the CDS is:

$$(1-\delta)\cdot(p_1 B(1) + q_1 p_2 B(2) + q_2 p_3 B(3) + q_3 p_4 B(4) + q_4 p_5 B(5))$$

- The value of Premium Leg assuming the fee of 1 bp is:

$$(B(1) + q_1 B(2) + q_2 B(3) + q_3 B(4) + q_4 B(5)) \cdot 1bp$$

- And the CDS fee is given by:

$$S_{RP} = 10^4 \cdot (1-\delta) \cdot \frac{\sum_{i=1}^{5} q_{i-1} p_i B(i)}{\sum_{i=1}^{5} q_{i-1} B(i)}.$$

- What is the catch? The p's (and, in turn, the q's) must also be backed out from bond prices under the Recovery of Par assumption. See Homework for an illustration.

4. Evolution of the Risky Term-Structure

- We combine the forward probabilities of default with the short rate tree to identify the evolution of the risky term-structure.

- At the end of Year 1, there are two states that can occur: where the riskless short rate is 14.32% and where it is 9.79%.

- Corresponding to these rates, there are two possible prices for the riskless bond:

$$B_u(1) = \frac{1}{1.1432} = 0.8747$$

and

$$B_d(1) = \frac{1}{1.0979} = 0.9108$$

Risky Term-Structure (Cont'd)

- Consider a one-year risky bond at this stage.
- Let $B_u^*(1)$ and $B_d^*(1)$ denote its prices in the states u and d, respectively.
- The one-year probability of default on this bond at this stage is 0.00892, as identified earlier.
- Therefore, the expected payoffs on this bond are

$$(1 - 0.00892) \cdot 1 + (0.00892) \cdot \delta \;=\; 0.99108 + 0.00892\delta.$$

- Thus, the expected return on the bond in the two states are:

$$\frac{0.99108 + 0.00892\delta}{B_u^*(1)} \quad \& \quad \frac{0.99108 + 0.00892\delta}{B_d^*(1)}.$$

Risky Term-Structure (Cont'd)

- The expected return in state u must equal the riskless one-year rate which is 14.32%.

- Solving for $B_u^*(1)$, we get $B_u^*(1) = 0.87006$.

- Similarly, the expected return in state d must equal the riskless one-year rate which is 9.79%.

- This gives us $B_d^*(1) = 0.90596$.

- Expressing these in terms of yields, the possible risky yields after one year are 14.935% and 10.381%.

- Equivalently, the possible values of the short spreads after one year are 0.615% and 0.591%.

Risky Term-Structure (Cont'd)

- Iterating, we can identify the spreads after two years, three years, etc.
- The final tree of short spreads (in %) has the following form:

Year 1	Year 2	Year 3	Year 4	Year 5
				1.003
			0.867	
		0.741		0.955
	0.615		0.826	
0.500		0.706		0.918
	0.591		0.796	
		0.681		0.890
			0.774	
				0.868

- In a similar fashion, by using riskless yields and risky bonds of longer maturities, we can also identify the evolution of risky yields of longer maturities.

5. Mark-to-market value (MTM) of a CDS

- Consider a default swap entered into a while ago at a spread S.
- Suppose the swap has T years *left* to maturity.
- Let the current price of a T-year swap be S_T.
- What is the marked-to-market value of the swap?
- This is a similar question to the marking-to-market of say an interest rate swap.
- As in that case, there are two strategies:
  - Either unwind the swap at the market CDS fee today for a par swap and value this portfolio.
  - Or value all outstanding cash flows of the current swap using market-implied probabilities of default.
- We illustrate the first approach below since second is a simple extension of the earlier valuation exercise.

MTM of a CDS (Cont'd)

- To close out the original position, we can take an offsetting position in a T-year CDS.

- For specificity, suppose we were buying protection in the original swap.

- To offset, we sell T-year protection.

- Resulting net cash flow: $S_T - S$.

- Value of swap:

$$[S_T - S] \times PV01$$

  where $PV01$ is the value of a 1 bp coupon stream which terminates at maturity of the swap or upon default whichever occurs first.

- $PV01$ is exactly the value of the Premium Leg we calculated earlier!

MTM of a CDS (Cont'd)

- From our previous analysis,

$$PV01 = \sum_{i=1}^{5} q_{i-1} B(i).$$

- Therefore, MTM of CDS is

$$[S_T - S] \times \sum_{i=1}^{5} q_{i-1} B(i).$$

- Example: Suppose in our Recovery of Treasury setting, we have an existing CDS with 5 years left to maturity, struck at 60 bp at time of origination.

- In our notation above, $T = 5$, so that $S_T = 46.56$ bp.

- Thus, MTM of this CDS (per notional) is

$$(46.56 - 60) \times 3.545 = -47.64 bp.$$

Identifying $PV01$ in practice

- How is $PV01$ identified in practice?
- Three simple steps:
  - Step 1: Make a recovery rate assumption.
  - Step 2: Using market data on CDS spreads, identify default probabilities in each period of the swap (under some reduced-form model).
  - Step 3: Using the default probabilities (and implied survival probabilities), compute $PV01$.
- For example, we can extract default probabilities from credit default swaps recursively using the Litterman-Iben model.
- This procedure is illustrated below.

Identifying Default Likelihoods from CDS Prices (RT assumption)

- What is par CDS fee for 1 year (in bp)?

$$S_1 = 10^4 \cdot (1 - \delta) \cdot p_1.$$

- Thus, S_1 and a recovery rate assumption imply p_1.

- What is par CDS fee for 2 year (in bp)?

$$S_2 = 10^4 \cdot (1 - \delta) \cdot \frac{(p_1 + q_1 p_2)B(2)}{(B(1) + q_1 B(2))}.$$

- Since we know p_1 and thereby q_1, S_2 implies p_2, and so on.

- Naturally, we need the risk-free term-structure for the entire extraction exercise.

- Which interest-rate curve should be picked in practice? The default now is the swap curve from which LIBOR discount factors are calculated.

Identifying Default Likelihoods from CDS Prices (RP assumption)

- What is par CDS fee for 1 year (in bp)?

$$S_1 = 10^4 \cdot (1 - \delta) \cdot p_1.$$

- Thus, S_1 and a recovery rate assumption imply p_1.

- What is par CDS fee for 2 year (in bp)?

$$S_2 = 10^4 \cdot (1 - \delta) \cdot \frac{(p_1 B(1) + q_1 p_2 B(2))}{(B(1) + q_1 B(2))}.$$

- Since we know p_1 and thereby q_1, S_2 implies p_2, and so on.

- Choice of a reduced-form model other than Litterman-Iben model alters the formulae, but does not significantly alter this overall approach.

6. Modeling Default Likelihood as Intensities: An Introduction

- In reality, default need not occur at the nodes of a binomial tree as in the Litterman and Iben model.
- Indeed, the exact timing of default is itself an important risk associated with default.
- To allow for this, we need to calibrate to market prices a more continuous measure of default likelihood.
- This may also be necessary to use default likelihoods from one set of instruments to price another set with differing maturities of cash flows.
- Next, we consider such continuous modeling through what are called as "default intensities."

Modeling Default Likelihood as Intensities (Cont'd)

- Default in the general reduced-form approach is handled either through an *intensity* process or through a ratings-based approach.
- We examine the former here; the latter is not as popular and hence skipped.
- In the intensity based approach, the fundamental process is an *intensity process* λ_t.
- The simplest case is that of a constant intensity: $\lambda_t = \lambda$ for all t.
- It is useful to first examine constant intensity processes in some detail to get a feel for the modeling process.
- To allow for a richer term-structure of credit spreads, non-constant but deterministic intensities or stochastic intensities are also considered in the literature.

Constant Intensity Default Processes

- In a constant intensity process, we have $\lambda_t = \lambda$ for all t, where $\lambda > 0$.

- The intensity (or *hazard rate*) λ has the following interpretation: the likelihood that a firm will survive at least t years is given by

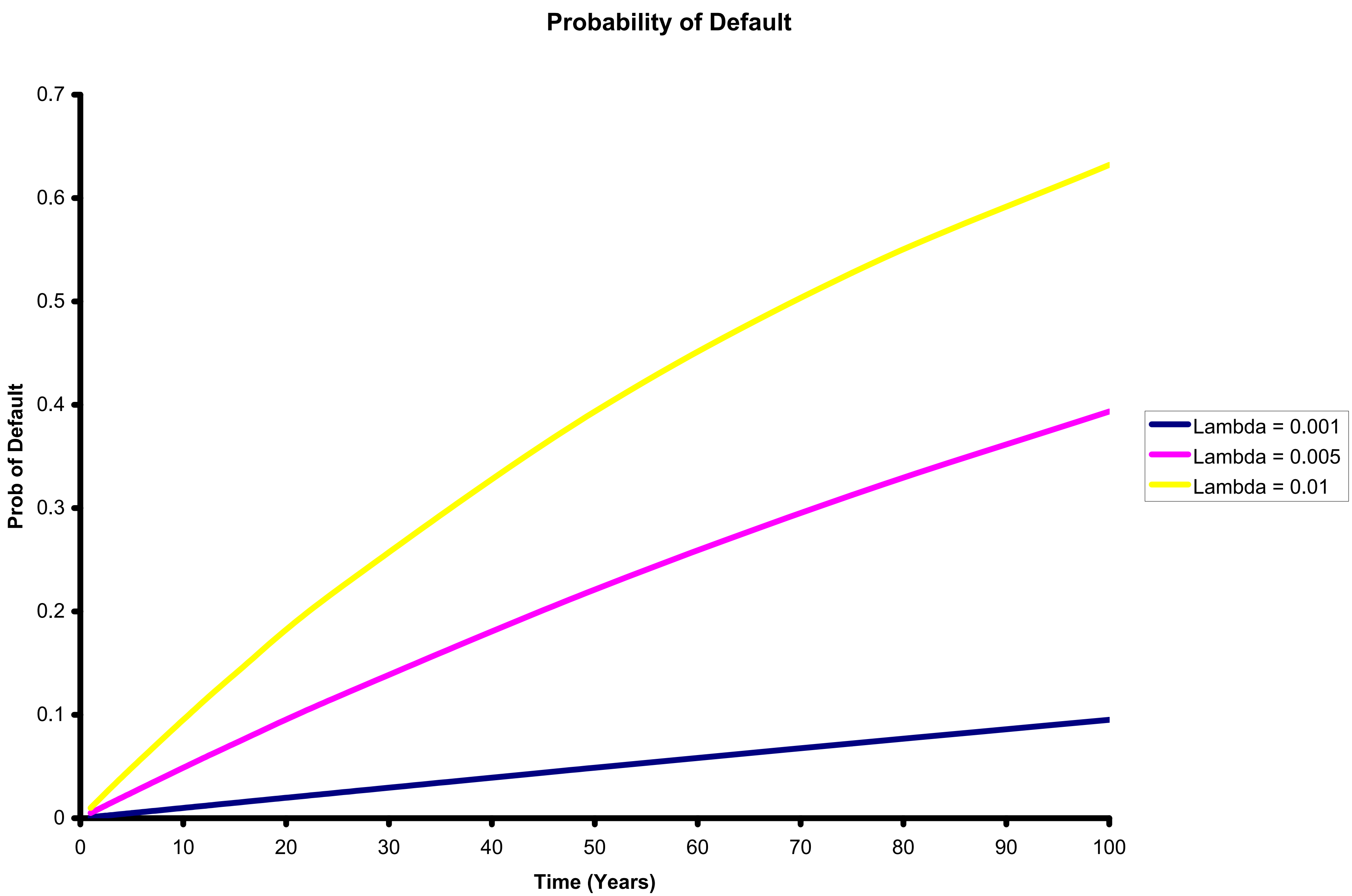
$$e^{-\lambda t}.$$

- In other words, the probability of default and no default are, respectively,

$$p = 1 - e^{-\lambda t} \qquad 1 - p = e^{-\lambda t}.$$

- A higher value of λ implies a lower probability of survival since $e^{-\lambda t}$ decreases as λ increases.

- The figure on the next page plots the survival probabilities for different values of λ.


Probability of Default
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
Prob of Default
0
10
20
30
40
50
60
70
80
90
100
Time (Years)
Lambda = 0.001
Lambda = 0.005
Lambda = 0.01

Litterman-Iben and Intensity-Based Modeling

- Question: What intensity process underlies Litterman-Iben? That is, what is λ_t?
- The underlying intensity has the form

$$\lambda_t = \begin{cases} a_1, & \text{if } t \leq t_1 \\ a_2, & \text{if } t \in [t_1, t_2) \\ a_3, & \text{if } t \in [t_2, t_3), \\ \vdots & \vdots \end{cases}$$

  where t_1, t_2, etc, correspond to the maturities at which we observe the zero prices (in the Litterman-Iben model, this is Years 1, 2, 3, 4, and 5).

- Why? Because the intensity process above implies that the likelihood of default is constant within a given period, but allows it to vary across periods, as is the case in the Litterman and Iben model.
- Note that essentially, Litterman and Iben model involves having as many free parameters as there are maturities, so the spread curve can always be matched exactly.

Litterman-Iben and Intensity-Based Modeling (Cont'd)

- In our example, we have to set

$$p_t = 1 - e^{-a_t \cdot 1}$$

since our each period is for one year.

- In other words,

$$a_t = -\ln[(1 - p_t)].$$

- For the case with Recovery of Treasury assumption, the final result is:

Year	**Probability of Default**	**Probability of Survival**	λ_t or a_t
1	0.00754	0.99246	0.007568
2	0.00892	0.98361	0.008960
3	0.01028	0.97350	0.010330
4	0.01178	0.96203	0.011849
5	0.01321	0.94933	0.013296

Litterman-Iben and Intensity-Based Modeling (Cont'd)

- What is the advantage of the intensity approach?

- Assuming constant intensity within a period, the RNP of default between time 0 and 0.5 years is simply

$$p_{0.5} = (1 - e^{-a_1 \cdot 0.5}) = 1 - e^{-0.007568 \cdot 0.5} = 0.00378.$$

- Similarly, the RNP of survival until 1.5 years is simply

$$q_{1.5} = e^{-a_1 \cdot 1 - a_2 \cdot 0.5} = e^{-0.007568 - 0.008960 * 0.5} = 0.988024.$$

- It turns out that a different recovery rate assumption (Recovery of Market Value) leads to a convenient way of expressing and understanding credit spreads in terms of default intensity and recovery rate.

- We will study this assumption and the model employing it (Duffie and Singleton, 1999) in some detail.

Constant Intensity Processes: Mathematical Context [OPTIONAL READING]

- Let $T_1, \ldots T_n$ denote the arrival times of some event (e.g., customers arriving at a post office).

- Suppose the *inter-arrival times* $T_{k+1} - T_k$ are independent and exponentially distributed: for some $\lambda > 0$, we have

$$\text{Prob}\,(T_{k+1} - T_k \leq \tau) \;=\; 1 - e^{-\lambda\tau}$$

- Then, the sequence (T_k) is called a homogeneous Poisson process with intensity λ.

- Alternatively, if N_t denotes the numer of arrivals in the interval $[0, t]$, then $N = (N_t)_{t\geq 0}$ is said to be a homogeneous Poisson process with intensity λ if the increments $N_t - N_s$ are independent and have a Poisson distribution with parameter $\lambda(t-s)$:

$$\text{Prob}\,(N_t - N_s = k) \;=\; \frac{e^{-\lambda(t-s)}[\lambda(t-s)]^k}{k!}.$$

Mathematical Context [OPTIONAL READING] (Cont'd)

- In the credit-risk setting, default is viewed as the *first jump time* of the counter N.

- Thus, the time of default is the distribution of the first arrival time T_1, which is exponential by assumption:

$$\text{Prob (Default before } t) \;=\; \text{Prob}\,(T_1 \leq t) \;=\; 1 - e^{-\lambda t}.$$

  This is the same thing as saying that the probability of the firm surviving past t is $e^{-\lambda t}$.

- The intensity λ is just the conditional default arrival rate:

$$\lim_{h \downarrow 0} \frac{1}{h} \text{Prob}\,(T_1 \in (t, t+h] \parallel T_1 > t) \;=\; \lambda.$$